THE

PHYSICAL SOCIETY

OF

LONDON.

PROCEEDINGS.

VOLUME XXXII.—PART I.

DECEMBER 15, 1919.

Price to Non-Fellows, **4s.** *net, post free* **4/3.**

Annual Subscription, **20/-** *post free, payable in advance*

Published Bi-Monthly from December to August.

LONDON:
FLEETWAY PRESS, LTD.,
1, 2 AND 3, SALISBURY COURT, FLEET STREET.

1919.

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I. *The Effect of Pressure and Temperature on a Meter for Measuring the Rate of Flow of a Gas.* By N. W. McLachlan, *D.Sc., M.I.E.E.*

Communicated by E. H. Rayner, M.A., Sc.D.
Received June 19, 1919.

A type of meter for measuring the rate of flow of a gas in litres per minute, or in any other unit of like dimensions, is shown diagrammatically in Fig. 1. It consists of a flat movable vane V (balanced and damped) on pivots, controlled by a spring S, the moving system being somewhat similar to that of a moving coil ammeter. The vane is deflected by the gas which issues from a nozzle N, and leaves the instrument case C by the exit tube E. The upper side of the case, the overall

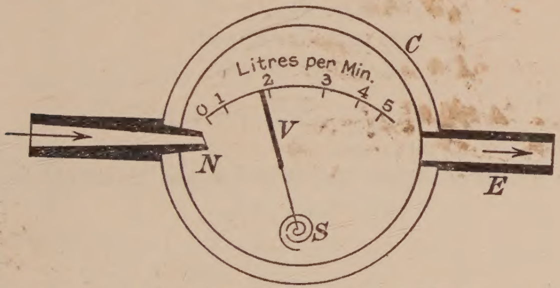


Fig. 1.—Diagrammatic Sketch of Meter.

diameter being about 5·5 cm., is covered with glass, and the construction of the instrument is arranged so that there are no leaks when it is in use.

1. *Theory of Instrument.*

The force normal to the movable vane of the meter will be assumed as

$$F=k\rho Av^n, \qquad (1)$$

where k=constant,
ρ=density of gas,
A=effective area of nozzle,
v=velocity of gas jet, and
n=an index, which is shown hereafter to be constant.

The value of k depends on the area of the vane, the angle it makes with the gas jet, and the influence of the instrument case. In the experimental work described herein, the lines of flow of the gas were parallel, and the jet did not break up before it impinged on the vane.

Case (*a*).—When the "mass of gas" (equivalent to the volume at normal temperature and pressure) passing through the meter per minute is constant for all exit pressures, and the vane is fixed in position, we have the force normal to the vane,

$$F = k(\rho A v)v^{n-1}$$
$$= k \times \text{mass per min.} \times v^{n-1}. \quad . \quad . \quad . \quad (2)$$

Also
$$\rho = \frac{\rho_0}{p_0} p \frac{273}{T},$$
$$= k_1 \frac{p}{T},$$

where
ρ_0=density of gas at N.T.P.,
p_0=760 mm. Hg.,
p=exit pressure of gas,
T=abs. tempr. of gas °C.,

and
$$k_1 = \rho_0 \frac{273}{p_0}.$$

Since $\rho A v$ is constant, and A is assumed constant, it follows that
$$\rho v = \text{const.} = k_2,$$

or,
$$v = \frac{k_2}{\rho} = k_3 \frac{T}{p}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where
$$k_3 = k_2 / k_1 = \text{const.}$$

From (3) it is clear that, provided the temperature is constant, the velocity of the gas increases with decrease in the pressure of the external atmosphere into which the gas is discharged. This pressure is taken to be equal to the static pressure of the gas jet. Since a decrease in the external pressure is equivalent to an increase in height in the isothermal atmosphere *—*i.e.*, an increase in altitude—the velocity of the gas jet increases with the altitude.

Substituting in (2) for v, we obtain

$$F = k \times \text{mass} \times \left(k_3 \frac{T}{p} \right)^{n-1}$$
$$= c \left(\frac{T}{p} \right)^{n-1}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

* In the work described below, the height of the isothermal atmosphere is taken. The relation between isothermal height and pressure is given by $h = 27{,}200 \log_e 760/p$ mm., where h is in feet. Ground level is taken as 760 mm. mercury and 10°C. The curve of Fig. 2 shows the relation graphically.

where c=a constant, when the vane occupies a definite position with respect to the nozzle.

From (4) it follows that, provided the vane is fixed with reference to the nozzle, and the temperature is constant, the force normal to the vane increases with decrease in pressure—*i.e.*, increase in altitude, although the mass of gas flowing through the meter per minute is constant. Hence, if the vane were free to move, it would be deflected beyond its position at ground level. With a given pressure, corresponding to a definite altitude, a decrease in temperature, which is experienced in the upper atmosphere, is accompanied by a decrease

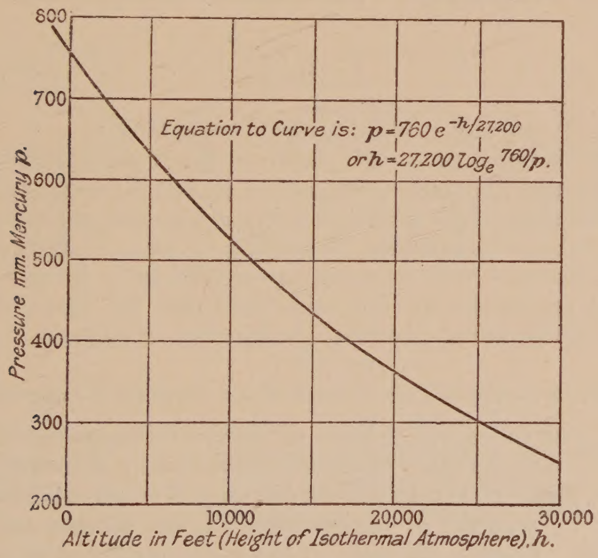


Fig. 2.—Curve showing Relation between Height of the Isothermal Atmosphere and the Pressure.

in the force on the vane. This is due to an increase in density and a consequent reduction in velocity. In this case the deflection of the vane will diminish. Thus, in the upper atmosphere where a reduction in both temperature and pressure is encountered, the effect of the former will partially compensate that of the latter.

Case (*b*).—When the meter reading is constant at all altitudes and temperatures, we have the force normal to the vane

$$F=k\rho Av^n=\text{a constant}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Substituting for ρ, $F=kk_1p/TAv^n$

$\therefore$ $v^n=c_1T/p$,

where c_1=a constant.

$\therefore$ $v^{n-1}=c_2(T/p)^{n'}$

where c_2=a constant,

and $n'=(n-1)/n$.

Thus, $(\rho Av)v^{n-1}=\text{mass}\times c_2\left(\frac{T}{p}\right)^{n'}$ = a constant; or mass $\propto (p/T)^{n'}$.

Since the mass of gas is proportional to the volume at N.T.P. we get

$$V=c_3\left(\frac{p}{T}\right)^{n'}, \qquad (6)$$

where V=volume at N.T.P., and c_3=a constant.

If n' is constant at all pressures, for a given flow-meter reading and constant temperature, it will be seen from (6) that the mass of gas decreases with the pressure, and therefore with increase in altitude. When the pressure is constant the mass increases with decrease in temperature. Moreover, as stated previously, in connection with case (a), the effect of temperature will partially compensate that of altitude.

2. *Experiments at Low Pressures and Constant Temperature.*

In order to test the validity of the above theory, the experimental work described below was carried out. Since the scale of the meter from 0 to 5 litres per minute is only about 4 cm. long, the accuracy of the readings was not all that could be desired. Further, the vane of the meter oscillated when the flow exceeded 5 litres per minute, so that little or no reliance could be placed on observations above this figure.

The apparatus used during the experiments is shown diagrammatically in Fig. 3. An oxygen cylinder with reducing valve is connected through a throttle valve to the flow-meter, which is situated in an air-tight compartment or chamber fitted with glass windows. The gas from the flow meter is passed through a wet meter, and discharged thence into the chamber. A vacuum pump is used to keep the pressure of the gas in the chamber equal to that corresponding to some definite altitude. The pressure of the gas in the chamber is

measured by means of a mercury manometer. In conducting a test at any particular altitude, the oxygen supply is adjusted so that the flow-meter reads a certain number of litres per minute—*i.e.*, the pointer is at a fixed mark on the scale. The control valve to the pump and the leak to the outer atmosphere are arranged to keep the pressure in the chamber constant. The time for two or more revolutions of the wet meter is taken. Knowing the time per revolution, the vapour pressure of the water in the meter, also the temperature and pressure of the gas in the chamber, the readings of the wet meter are reduced to normal temperature and pressure—*i.e.*, 0°C. and 760 mm. Hg. The calibration of the flow-meter obtained in this way is correct only when the gas used is oxygen at the same tem-

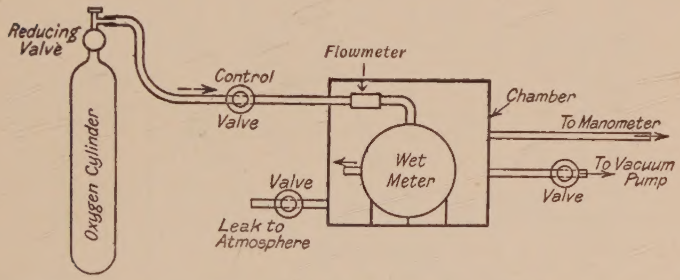


Fig. 3.—Diagram of Apparatus used for Experiments at Low Pressures.

perature and pressure as that of the calibration test. If the meter is used to measure the rate of flow of air, or any other gas, it will have to be recalibrated for the gas in question, or a correction made for the different density thereof, as shown in the appendix.

3. *Results of Experiments at Low Pressures and Constant Temperature.*

The results of the experiments are illustrated graphically in Figs. 4, 5, 6 and 7. The calibrations of the meter at ground level (in this case −670 ft. = −200 metres on the isothermal atmosphere scale) and at various altitudes up to 30,000 ft. = 9,000 metres, are given. It appears from the results, that the l ation between the N.T.P. volume of gas per minute and the ow-meter reading is very nearly linear at any altitude.

The relation between N.T.P. litres per minute and the altitude in feet for various flow-meter readings is shown in Fig. 6. The set of points for any particular flow-meter reading do not lie on a straight line, but on a curve of the form $y=ae^{-bx}$.

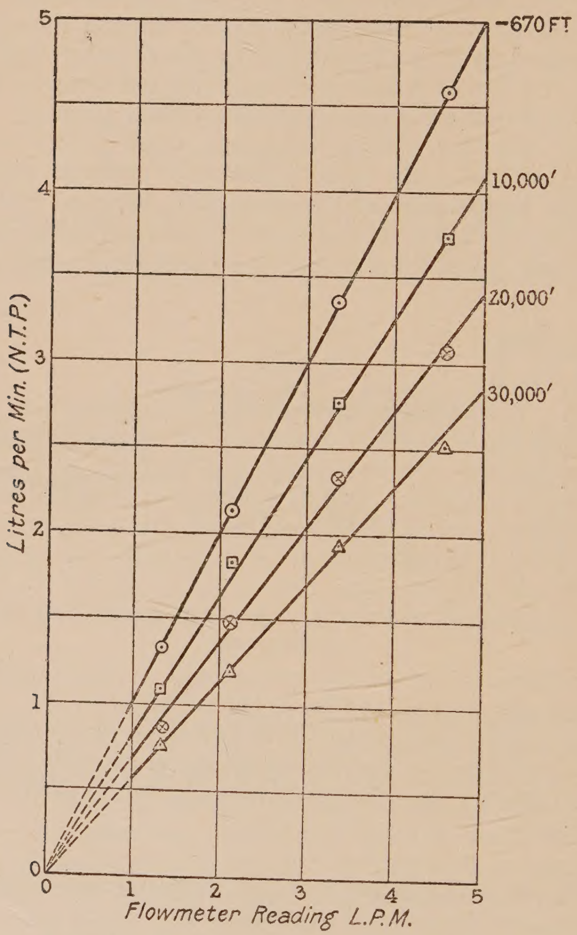


FIG. 4.—CALIBRATION OF METER AT VARIOUS ALTITUDES (EXPERIMENTAL).

The reason for this is outlined in the appendix. It may be sufficiently accurate, for practical purposes (since the accuracy of the meter itself is not great owing to its size), to draw the best straight lines through the sets of points. The slopes of

the lines thus obtained diminish with the meter reading. This is due to a diminution in the value of c_3 in equation (6). When the above procedure is adopted, we may write

$$V+mh=V_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

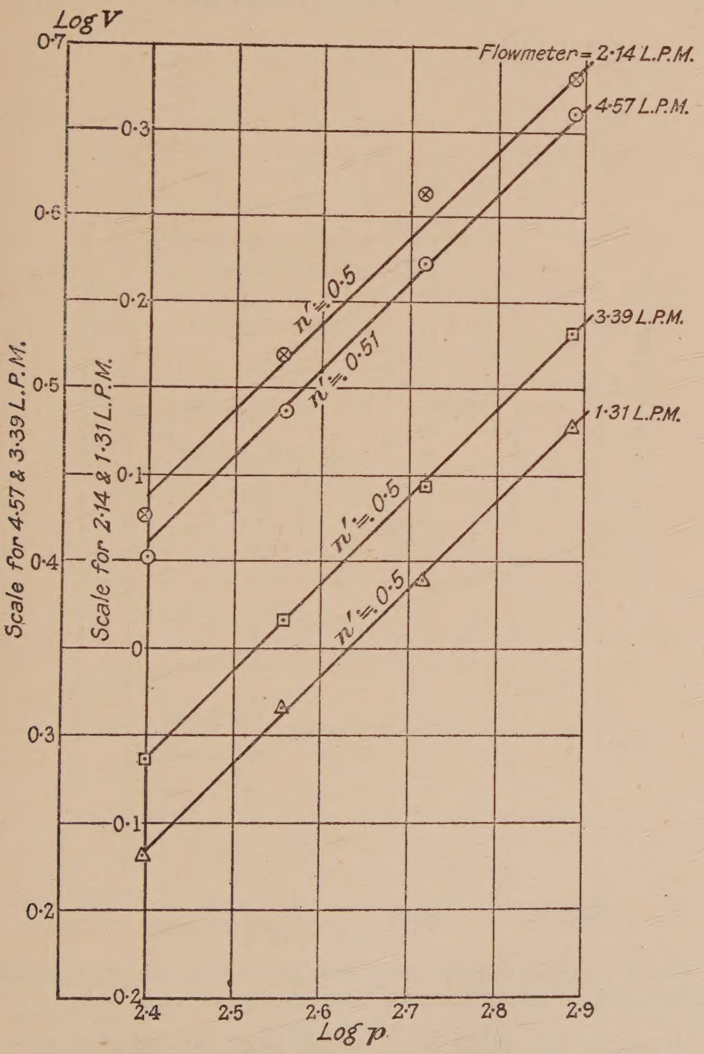


FIG. 5.—LOGARITHMIC PLOTTING OF N.T.P. VOLUME IN LITRES PER MINUTE, AND PRESSURE MM. OF HG. FOR VARIOUS METER READINGS.

where V_0=N.T.P. volume at ground level,
h=altitude in feet,
m=slope of line,
and V=N.T.P. volume at altitude h.

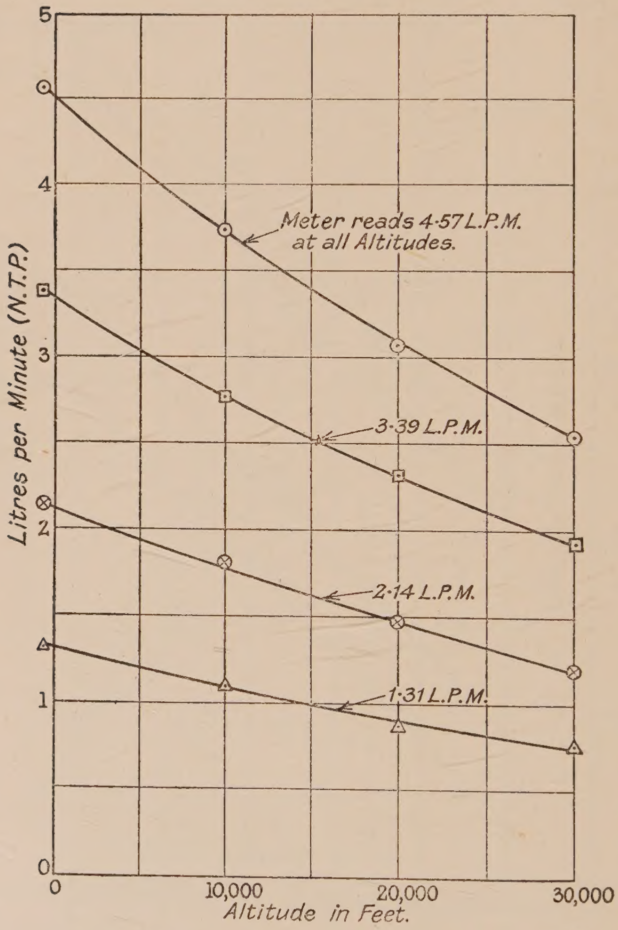


FIG. 6.—CURVES SHOWING RELATION BETWEEN N.T.P. VOLUME AND ALTITUDE FOR VARIOUS METER READINGS (EXPERIMENTAL).

Knowing the value of m, which must be regarded as an average for the particular set of points for a definite reading of the meter at all altitudes, the N.T.P. volume of gas at *any* altitude can be calculated.

In Fig. 7 the results of Fig. 6 are plotted on a pressure base, which causes an alteration in the curvature from convex upwards to concave upwards.

Log V is plotted against log p in Fig. 5. The sets of points for 1·31 and 3·39 litres per minute are almost collinear, but

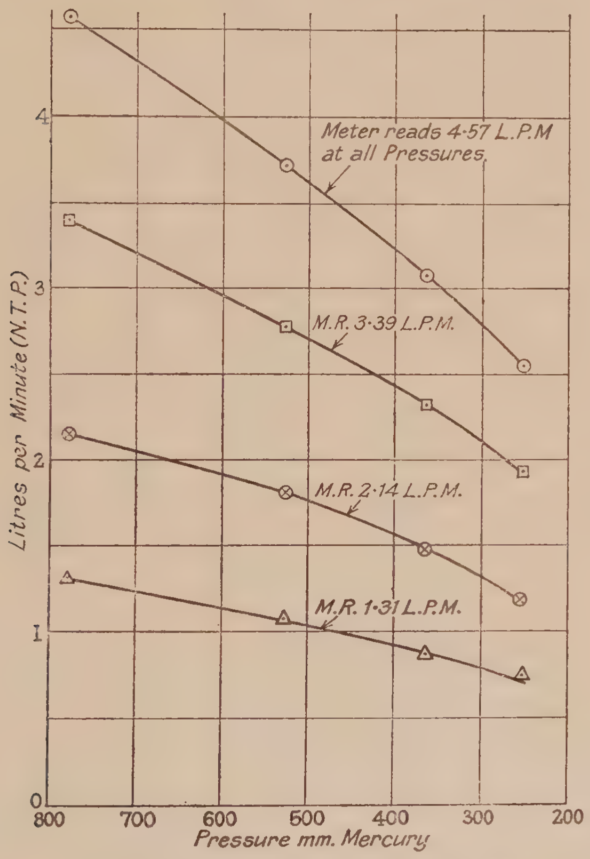


Fig. 7.—Curves of Fig. 6 on a Pressure Base.

those for the other flow-meter readings lie on curves having a slight concavity upwards. This may be due to experimental errors. Taking the points to lie on straight lines in all cases, the slopes of these lines represent the values of n' in equation (6).

Using the results obtained from Fig. 5, the value of n' may be taken as 0·5 on the average. The corresponding value of

n in equation (1) is 2. Thus the force on the vane, when the latter occupies a definite position with reference to the nozzle, varies approximately as the square of the velocity of the gas impinging on it. The slight variations in n' may be caused by the influence of the case, and also by interference effects due to the position of the vane with reference to the nozzle and to the exit pipe.

TABLE I.—*Meter Reading Throughout*=4·57 *L.P.M.*

N.T.P. litres per minute.		Altitude in feet (isothermal atmosphere).	Metres.
Experiment.	Calculation, assuming $n'=0{\cdot}5$		
4·57	4·57	−670	−200
3·73	3·76	+10,000	3,000
3·07	3·12	+20,000	6,000
2·53	2·6	+30,000	9,000

$T=11$°C, barometer=779 mm. Hg.

TABLE II.—*Meter Reading Throughout*=2·14 *Litres per Minute.*

N.T.P. litres per minute.		Altitude in feet (isothermal atmosphere).	Metres.
Experiment.	Calculation, assuming $n'=0{\cdot}5$		
2·14	2·14	−670	−200
1·81	1·76	+10,000	3,000
1·47	1·46	+20,000	6,000
1·19	1·22	+30,000	9,000

$T=11$°C, barometer=779 mm. Hg.

Tables I. and II. have been inserted to show the agreement between experiment and calculation. The calculated values have been obtained from the ground level readings of the meter by the method given in the appendix. The index n' in equation (6), on which the calculations are based, has been taken as 0·5. It will be seen that the discrepancy between experiment and calculation is never large. The greatest error occurs in the second row of Table II.—viz., 1·81—and is probably an experimental one.

4. *Experiments at Low Temperatures and Constant Pressure.*

The apparatus used for these tests is shown diagrammatically in Fig. 8. The gas from an oxygen cylinder passes through throttle valve (1) and thence to meter A. The water manometer is used to measure the fall in pressure between meters

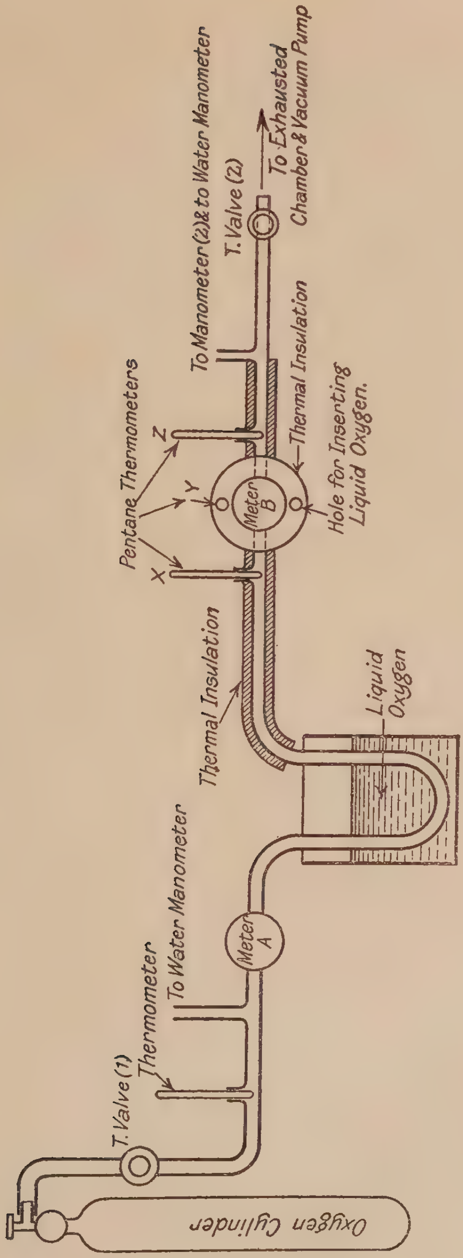


Fig. 8.—Diagram of Apparatus used for Low Temperature Tests.

A and *B*. Manometer (2) is used to read the pressure of the gas issuing from *B*. During the experiments, which were conducted at ground level, the pressure drop *AB* was always less than 3·5 cm. of water (about 2·5 mm. Hg.), so that it is sufficiently accurate to assume that the pressure of the gas is the same at both meters. Care must be taken in connecting the manometers to the main flow pipe. When T pieces are used they must be free from internal projections. A projection or lip on the connection to the main pipe will cause either suction or pressure, according to its position with reference to the gas stream. This will affect the manometer accordingly.

After passing through *A*, the gas traverses a U tube, thermally insulated from *A*, immersed in liquid oxygen, which produces a reduction in temperature. Thence it passes through *B*, which is situated in an enclosure packed with cotton wool or other insulating material, to an exhausted chamber, the pressure in which is kept approximately constant. If the flow-meter is to be tested at atmospheric pressure, the chamber and throttle valve (2) can be disconnected. By means of the apparatus of Fig. 8 it is possible to carry out experiments in which the gas pressure can be fixed at 760 mm. Hg., or at values greater or less than this. This is effected by manipulation of the throttle valves. In working at pressures different from that of the external atmosphere great care must be taken to make the whole of the apparatus leak free. The temperature is controlled by altering the length of the U tube in the liquid oxygen. It depends largely on the mass of gas flowing through the tube per minute. The larger the mass of gas up to a certain limit the lower the temperature of the gas passing through meter *B*. When the system is thoroughly cooled, the influence of the mass of gas is considerably less. The temperature control may be facilitated by an additional U tube immersed in water, which can be heated if necessary, or in liquid oxygen, according to requirements. It is necessary, when readings are taken at constant temperature, to wait until conditions are steady. The temperature of the gas should be equal on both sides of flow-meter *B*. To secure this condition it is essential that the flow-meter and the connections thereto, should be well insulated thermally. It was found necessary to pour liquid oxygen into the insulation surrounding the meter in order to attain the desired temperature condition. Before the flow-meter was insulated, thermometer *X* regis-

tered 95°C. and thermometer Z 10°C. The temperature of the gas issuing from the nozzle, however, was almost 95°C., and the results were not so erroneous as might have been expected. In experiments of this nature it is inadvisable to use air, owing to the solidification of the moisture therein, which chokes the system and causes a large pressure drop between meters A and B, apart from other effects. Some experiments were conducted by replacing the oxygen cylinder and valve (1) by a calcium chloride drying tower open at one end to the atmosphere. These were not satisfactory for the above reason. In this case a correction must be made for the drop in pressure in the chloride tower.

When an experiment is being performed with the above apparatus the temperature of the gas passing through flow-meter B is adjusted to a value, T_2, and the flow regulated so that the flow-meter gives a definite reading. The pressure at meter B is adjusted to a certain value, say the atmospheric pressure at the time, by manipulating the throttle valves. The reading of manometer (1) should be less than 3·5 cm. of water. When conditions are steady, readings are taken of T_1, flow-meter A, T_2, flow-meter B and the barometric height. The calibration of flow-meter A under conditions almost identical with those during the test is found by disconnecting the whole of the apparatus to the right of A, and substituting a wet meter, the gas from which is discharged into the atmosphere. A correction can be made for any variation in conditions, as shown hereafter. The readings of the wet meter are reduced to N.T.P. These represent the N.T.P. volumes of gas passed through flow-meter B during the experiments. It was found more accurate to mark the positions of the pointer of flow-meter A on a paper scale pasted to the glass than to read A directly, since the position of the needle during calibration could be obtained more accurately. Meter B is then calibrated in a manner similar to A, when the pointer indicates the same readings as those obtained at low temperatures. It is really immaterial, for purposes of comparison, under what conditions A and B are calibrated at ground level, so long as they are identical and the temperature is known.

5. *Results of Experiments at Low Temperatures and Constant Pressure.*

Some of the results obtained are shown in Table III. It appears that the agreement between experiment and calcula-

tion is better for the two higher than for the two lower readings. Considering the liability to errors of observation at the lower part of the instrument scale, there is a fair agreement between theory and experiment.

TABLE III.

$T=13°C.$, barometer=761 mm. Hg.

Temperature, T_3(°C.).	Reading of meter B (virtual litres per minute).	N.T.P. litres per minute passing through meters A and B.	
		Experiment.	Calculation, assuming $n'=0{\cdot}5$
−100	5·13	6·55	6·56
−90	4·18	5·28	5·22
−80	3·14	3·90	3·83
−70	2·33	2·82	2·77

6. *Experiments at Increased Pressures and Constant Temperature.*

From equation (6) viz., $V=c_3(p/T)^{0\cdot5}$, it is evident that the effect of increasing the pressure of the gas passing through the flow-meter, is equivalent to lowering the temperature, since both cause an increase in density. Similarly an increase in temperature is equivalent to a decrease in pressure, the density in this case being lowered. In view of this, some experiments were carried out at increased pressures and constant temperature. The apparatus was arranged as shown in Fig. 9. The increase in pressure above that of the external

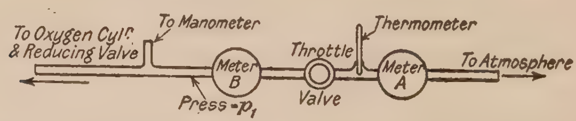


FIG. 9.—DIAGRAM OF APPARATUS USED FOR INCREASED PRESSURE TESTS.

atmosphere is measured by a mercury manometer, and the reduction in pressure before the gas passes through A is effected by a throttle valve. As in the experiments at low temperatures, the reading of B is observed and the corresponding position of the pointer of meter A is marked on paper pasted to the glass top. The calibrations of both meters are determined as before. In order to avoid a difference in temperature of the gas at the two meters, due to expansion at the throttle valve, it may be necessary to use a heating

tube between the throttle valve and meter *A*. The heating tube was not required in the present case, since the temperature was equal on the high and low pressure sides of the system, due to heat obtained from the connecting pipe, &c. In these experiments care must be taken to avoid leaks on the high pressure side of the system.

7. *Results of Experiments at Increased Pressures and Constant Temperature.*

TABLE IV.

T=13°C., barometer=760 mm. Hg.

Pressure (p_1 mm. Hg.)	Temperature which produces same change in density as p_1 (°C.).	Reading of *B* (virtual litres per min.).	N.T.P. litres per minute passing through meters *A* and *B*.	
			Experiment.	Calculation.
1,255	−100	5·13	6·55	6·56
1,188	−90	4·18	5·28	5·22
1,125	−80	3·14	3·88	3·83
1,069	−70	2·33	2·72	2·77

The results are given in Table 4. The agreement between experiment and calculation is of a similar order to that obtained in the tests at low temperatures.

The meter readings for corresponding temperature and pressure, *i.e.*, equal gas densities, are tabulated in Table V. in order to show the agreement between the two sets of observations, both of which were made under practically the same atmospheric (external) conditions.

TABLE V.

T=13°C., barometer=760 mm. Hg.

Virtual litres per minute on meter *B*.	N.T.P. litres per minute, the gas density being the same for both sets of figures (experiment).	
	Increased pressure.	Reduced temperature.
5·13	6·55	6·55
4·18	5·28	5·28
3·14	3·88	3·90
2·33	2·72	2·82

Experiments have not been carried out at low temperatures and low pressures (simultaneous conditions), nor under various other conditions possible. Unless extreme temperatures and

pressures are used there is no good reason to doubt the validity of equation (6), under various conditions of temperature and pressure.

In conclusion it should be mentioned that the instrument described herein behaves quite well when fitted to an aeroplane. The accuracy required in practice is not greater than 5 per cent., and the readings are certainly more accurate than this. The vibration of an aeroplane when in flight has little influence on the accuracy. The reading is affected when the aeroplane has an angular acceleration about an axis parallel to that of the spindle of the instrument movement. It is not likely to be read, however, when the acceleration is large.

8. *Summary of Results.*

The chief results may be summarised as follows :—

1. The force normal to the flow-meter vane, for a limited angular movement, at various temperatures and pressures which prevail in the upper and lower strata of the atmosphere, is given by $F=k\rho Av^2$. The value of k varies with the position of the vane relative to the nozzle.

2. A flow-meter will only read correctly if the temperature pressure and density of the gas is the same as that used during calibration. A correction can be made, however, for variation in temperature and pressure by means of the formula, $V=V_0(T_0/p_0 \cdot p/T)^{0 \cdot 5}$, where V=N.T.P. volume of gas under new conditions, V_0=meter reading*; T_0 and p_0=calibration temperature and pressure; p and T=new temperature and pressure. When corrected for a gas of different density the formula becomes $V=V_0(\rho/\rho_0 \cdot T_0/p_0 \cdot p/T)^{0 \cdot 5}$. (*See* Appendix for significance of symbols.)

Appendix.

Relation between Altitude and N.T.P. Volume of Gas per minute for a Definite Meter Reading.

The relation between the height of the isothermal atmosphere and the pressure is given by

$$h=a \log_e b/p,$$

where h=height, p=pressure, a and b=constants.

$$\therefore \qquad p=be^{-h/a}. \qquad (8)$$

* This is N.T.P. litres when the temperature of the gas is T_0 and its pressure p_0.

From (6) $V=c_3(p/T)^{n'}$, where V=N.T.P. volume.
Equating the values of p from (6) and (8) we get

$$V=\frac{d}{T}e^{-fh} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where d=a constant,

and $f=n'/a$.*

If n' and T are constant for a given reading of the meter, the relation between V and h is an exponential one [from (9)]. When, for all values of h, the product fh is small compared with unity, the relation between V and h is approximately linear. This can be shown by expanding e^{-fh} thus :—

$$V=\frac{d}{T}e^{-fh}=\frac{d}{T}\{1-fh+f^2h^2/2\,!-f^3h'^3/3\,!+f^4h^4/4\,!-, \text{\&c.}\}.$$

If fh=0·1, say, all the terms beyond the second are negligible, and we obtain

$$V=\frac{1}{T}(d-dfh), \text{ approximately,}$$

or

$$V+mh=\frac{d}{T} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where $m=\frac{df}{T}$. (*See* (7).)

Taking the average value of n' as 0·5, and the greatest value of h as 40,000 feet, $fh=\frac{0{\cdot}5\times 40{,}000}{27{,}200}=0{\cdot}74$. In this case the first five terms of the above series are important, and the relation between V and h, over the range h=0 to h=40,000 feet=12,200 metres, cannot be taken as linear. The range from ground level, over which (10) is applicable is 7,000 feet=2,130 metres.

If, for a definite meter reading, the value of n' varies with the altitude, it is evidently some function of h. Thus we may write $V=\frac{d}{T}e^{\varphi h}$, where $\varphi(h)$ is a function of h. It is probably sufficiently accurate for practical purposes to assume that n' is constant.

* $f=1{\cdot}84\times 10^{-5}$ when h is in feet, and $5{\cdot}56\times 10^{-5}$ when h is in metres.

Correction of Meter Reading at Various Altitudes.

In (9), when $h=0$, $V=V_0=d/T$=meter reading at the ground. Since $T=283°$abs., $d=283\ V_0$, and hence

$$V=283 V_0 \,.\, e^{\cdot fh}/T. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

If there was no variation in V with altitude and temperature, equation (11) would read $V=V_0$. Therefore, the quantity $283e^{\cdot fh}/T$, may be regarded as the correction factor of the meter corresponding to altitude h and absolute temperature T (it is assumed that V_0 corresponds to the N.T.P. volume when calibration is effected at 10°C. and 760 mm. Hg.). The absolute temperature can be written in terms of h (approximately), thus: $T=283-1{\cdot}75\times10^{-3}h$, where $1{\cdot}75\times10^{-3}$ is the temperature gradient per foot of ascent. Equation (11) now becomes

$$V=\{283 V_0/(283-1{\cdot}75\times10^{-3}h)\} \,.\, e^{\cdot fh}=V_0 y, \quad . \qquad (12)$$

where $y=\{283/(283-1{\cdot}75\times10^{-3}h)\} \,.\, e^{\cdot fh}$,
=correction factor.

Curves are given in Fig. 10, showing the correction factor at various altitudes (*a*) when the temperature is constant, *i.e.*, in the isothermal atmosphere; (*b*) when the variation of temperature with altitude is taken into account. The difference between curves (*a*) and (*b*) is due to the compensation caused by reduction in temperature. In curve (*b*) an average value of 1·75°C. has been taken as the fall in temperature per 1,000 ft.=300 metres of ascent. Fig. 11 shows the curves of Fig. 10 on a pressure base.

Correction of Meter Reading Due to Variation in Temperature and Pressure at Ground Level or at any Altitude.

From (6), $V=c_3(p/T)^{0{\cdot}5}$; where V=N.T.P. volume at pressure p and absolute temperature T. When $p=p_0$, $T=T_0$ and $V=V_0$=meter reading, these being conditions during calibration.

Thus, substituting these values in (6), we find

$$c_3=V_0(T_0/p_0)^{0{\cdot}5};$$

$$\therefore \qquad V=V_0(T_0/p_0 \,.\, p/T)^{0{\cdot}5}, \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

where V=N.T.P. volume.

By means of (13) it is possible to correct the meter reading to any absolute temperature T and pressure p.

Example of the Use of Correction Factor.

In the curves of Figs. 10 and 11 it has been assumed that the temperature of the gas used for calibration purposes is 10°C.

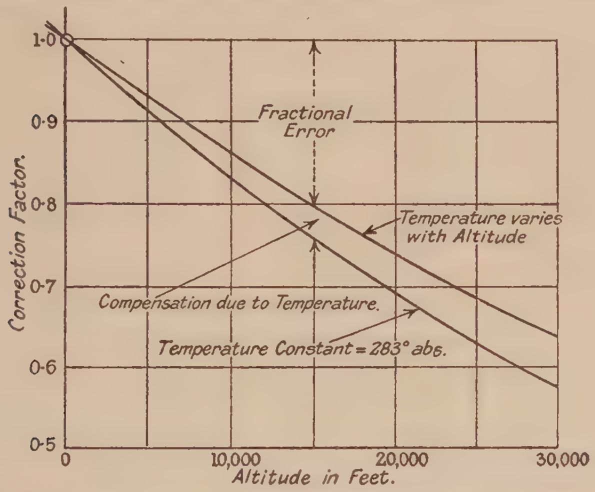


FIG. 10.—CURVES SHOWING CORRECTION FACTOR OF METER AT ANY ALTITUDE.

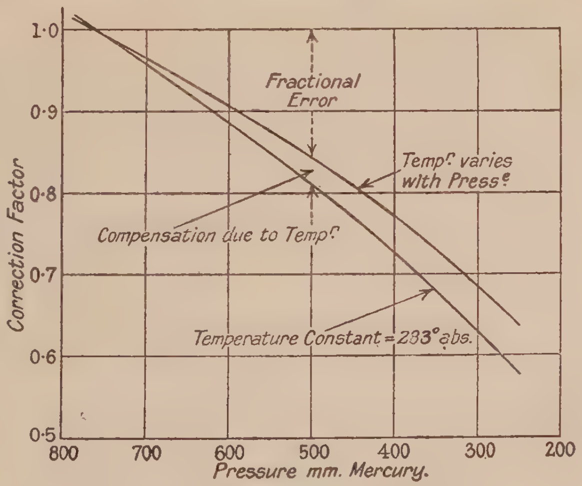


FIG. 11.—CURVES OF FIG. 10 ON A PRESSURE BASE.

and its pressure 760 mm. of mercury. These conditions will not be attained in general. Suppose the meter reads 4 litres per minute when calibration conditions are such that $T_0 = 15°C$.

and $p_0=737$ mm. Hg., *i.e.*, the N.T.P. volume of gas passing through the meter per minute under these conditions is 4 litres. At 10° C. and 760 mm. Hg. the N.T.P. volume of gas when the meter reads 4 L.P.M. is $4\left(\frac{288}{737}\times\frac{760}{283}\right)^{0\cdot5}$ $=4\times1\cdot025\doteqdot4\cdot1$ litres per min. When the meter reads 4 at, say, 30,000 ft. $=$ 9,000 metres, the N.T.P. volume of gas is ($4\cdot1\times$correction factor)$=4\cdot1\times0\cdot64\doteqdot2\cdot6$ litres per minute. The above procedure is necessary if the correction factors of Figs. 10 and 11 are to be used properly. If the reading of the meter had not been corrected to 10°C. and 760 mm. Hg., the error would have been 2·5 per cent.

Correction of Meter when Used for Gases of Different Densities.

When a gas other than that used during the calibration is passed through the meter, the reading can be corrected in the following manner :—

From (13) we have $V=V_0(T_0/p_0\ p/T)^{0\cdot5}$ the density of the gas at T_0 and p_0 being ρ_0. If, however, the density of the gas flowing through the meter at T_0 and p_0 is ρ, the true value, *i.e.*, the N.T.P. value, of V_0 is $V_0\,(\rho/\rho_0)^{0\cdot5}$. Whence

$$V=V_0(\rho/\rho_0\,.\,T_0/p_0\,.\,p/T)^{0\cdot5},$$

where V_0 is the reading of the meter.

ABSTRACT.

The theory of an instrument for measuring the rate of flow of a gas is outlined, the effects of variation in the temperature and pressure of the gas being taken into consideration. This theory is tested experimentally for pressures varying from 1,250 to 250 mm. Hg., and for temperatures from 10°C. to 100°C. It is found to be fairly accurate. The results are applied to the measurement of the rate of flow of gas on an aeroplane in the upper atmosphere, where a reduction in temperature and pressure is encountered. It is shown that the instrument reading for a certain N.T.P. volume of gas depends on the altitude, but that this volume can be obtained by using a correction factor.

DISCUSSION.

Mr. C. C. Paterson asked if the results depended on the structure of the meter, since the Paper only referred to one type.

Dr. A. Griffiths suggested that some automatic method of compensating the error would be advantageous as a direct reading instrument was always desirable.

The Author, in reply, said that compensating devices were under investigation when the war came to an end. The pressure effect could be compensated by using an aneroid control to vary the area of the nozzle, but he did not see how the temperature effect could be easily corrected. As regards the structure of the meter, the diameter of the nozzle was about 3/32nds of an inch, and the rectangular vane was mounted very similarly to the moving parts of a moving coil ammeter. Either electromagnetic damping or, for laboratory purposes, ordinary fluid damping could be used.

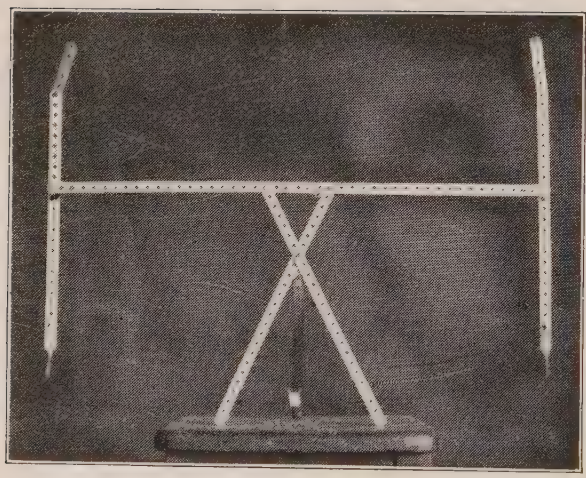


FIG. 1.—COMPLETE BALANCE, FILTER PAPER IN POSITION.

To face p. 21.

II. *A Cheap and Simple Micro-Balance.* By Capt. J. H. Shaxby, *B.Sc., University College, Cardiff.*

Received, July 11, 1919.

The name micro-balance should, perhaps, hardly be applied to the instrument described below, as it is designed for maximum loads varying between 5 milligrams and 5 decigrams. It is a simplified and rough form of a piece of apparatus designed for smaller loads, to be used in a research which the war has postponed.

The present type was put together for bacteriological use, to weigh small drops of fluid absorbed in bits of filter paper, rapidly and with an accuracy of one or two parts in 1,000. The balance was required to be reasonably robust and portable, and above all cheap. It has proved suitable for its purpose, and also in weighing small samples of mixed powders.

The principle is very simple : A light thread has its extremities attached to the lower ends of a pair of nearly vertical rods 2 ft. or so apart, each of which is pivoted at a point very little above its centre of mass. Thus, a very light load suspended at the middle of the thread causes a considerable depression of that point. The depression is not quite proportional to the load ; but a series of five weights, one-fifth, two-fifths, &c., of the maximum load, gives means of calibration.

Detail of construction is illustrated in the diagrams, and is as follows : The metal parts are built up of "Meccano" strips of various lengths, fixed together where necessary by the small nuts and bolts supplied with the sets. The fixed framework consists of a long horizontal arm, 2 ft. in length, built up of two 25-hole strips, with three holes overlapping, supported at a height of about 1 ft. by a sort of tripod. This is built up of three 25-hole strips (as shown in the photograph, Fig. 1), two of which are in a vertical plane, while the third, bent slightly about the third to fifth holes from the top, slopes away like the back leg of some easels. The lower end of this leg may be bent so as to lie flat on the table, to which a drawing-pin through the last hole clamps it rigidly and holds the whole frame steady.

To the ends of the horizontal beam are fixed short pieces bent twice at right angles, so that they lie as shown in the plan of the end of the beam (Fig. 2).

Each of the two vertical suspended arms consists of two strips, the lower a 25-hole one and the upper a 6-hole one, bolted together through their end holes. The purpose of this is to allow of ready variation of sensitiveness, as explained later. A short piece ($\frac{1}{4}$ in. long) of glass tubing with rounded-off ends is cemented into the 15th hole from the free (lower) end of the longer strip; this hole is slightly below the centre of

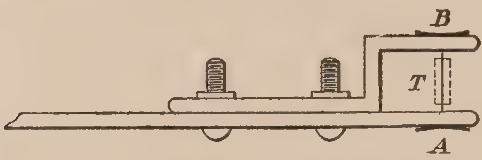


Fig. 2.—Plan of End of Fixed Beam.

mass of the compound strip. A small nut and bolt a few holes lower brings the centre of mass down sufficiently and permits of adjustment. The middle points of two short bits of copper wire are joined by a thread of ordinary sewing silk, so that the free length of thread is about equal to the distance AB (Fig. 2). One bit of wire is then passed through the end hole at A, through one of the glass tubes, T, and through the end hole at B (bent temporarily toward A), so that the wires now lie flat on the outer sides of A and B, and keep the thread taut.

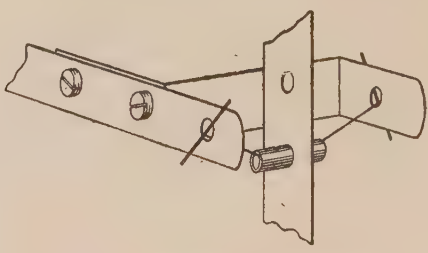


Fig. 3.—Diagram of Suspension.

The movable arm is thus suspended (Fig. 3) with very little friction. The other arm is supported similarly at the other end of the fixed horizontal beam. Light aluminium vanes attached to the lower ends of the vertical arms play in water dashpots and keep the whole dead beat.

A thin wire hook is also soldered on near the bottom of each arm (Fig. 4). Thin wire loops are attached to the ends of an unspun silk or other fine fibre of length equal to the distance

between the pivots of the movable arms, the loops are slipped over the hooks and the lower ends of the movable arms thus linked together. The tension in the thread, and thus the sensitiveness of the balance, can be largely and immediately varied by changing the angle between the two strips which make up each compound movable arm. The parts of the

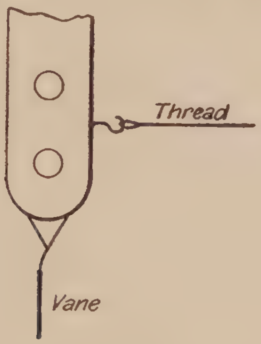


Fig. 4.—Lower End of Movable Arm.

balance will not in general be quite symmetrical, so that if one arm has its two parts in a straight line the other will have to be adjusted slightly, in order that the lower portions may rest at the same small angle with the vertical, *i.e.*, so that the middle point of the fibre shall lie directly under the middle point of the fixed horizontal beam. To decrease the sensitiveness, both the upper short strips are to be turned inward. If they are turned outward the fibre is usually no longer kept taut, so

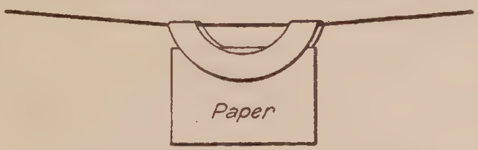


Fig. 5.—Clip and Filter Paper.

greater sensitiveness cannot be obtained thus ; if it is necessary, the small adjusting nuts are raised nearer to the pivots.

The loads are supported by a very light aluminium-foil clip loosely looped over the middle of the fibre ; the scrap of filter paper is inserted in this (Fig. 5). For powders a small thin paper scalepan replaces the clip.

Weighings are made thus. The calibrating weights are inserted in turn, and the resulting depressions noted. In the

case of weighing a drop the filter paper is included in all the observations. The depression due to the object to be weighed is finally observed, and its weight readily found from a calibration curve.

Of course the deflections could easily be measured by a cathetometer microscope, but as has been said above, cheapness was a *sine qua non*. The balance itself costs about eighteen-pence, and an auxiliary microscope costing several pounds was clearly ruled out. The deflections were measured as follows: A cardboard slider carrying a zero mark, and pierced by a circular "peephole" about the size of the pupil of the eye is attached to a millimetre scale, which is set up vertical 2 ft. or 3 ft. in front of the balance. A small cardboard tube closed at one end (the cap of an old rather wide thermometer case) was blackened inside, and across its mouth was stretched a fine fibre, which thus showed up sharp and white against the dark background of the interior of the tube. This was put an inch or two behind and slightly lower than the middle of the long balance fibre. If both fibres are horizontal a position can be found for the slider on the scale, so that one fibre hides the other from an eye looking through the peephole, when the clip is unloaded, and for each successive load. The deflections thus measured on the vertical scale are clearly equal large multiples of the actual deflections of the middle of the balance fibre. In practice it is found that the setting of the slider can be made with the greatest certainty, not with a horizontal fibre or a pair of cross-hairs on the tube, but with a single fibre set at a slight angle with the horizontal. The slider is adjusted so that the horizontal part of the balance fibre appears to bisect the short fibre in the tube. The angle between the two fibres should be small, and the slider can then be set with great precision; an astonishingly small displacement noticeably upsets the bisection.

ABSTRACT.

The instrument, devised for bacteriological use, had to be cheap and moderately robust. It consists of a long horizontal fibre joining the lower ends of two vertical beams, each pivoted very little above its centre of mass. A small weight acting at the middle of the fibre thus causes a considerable depression. This is read off by arranging a slider on a vertical millimetre scale about 2 ft. in front, so that the middle of the fibre and a second short fibre placed just behind it are in line with a "peephole" on the slider. Adjustment is provided for quickly and largely altering the sensitiveness. The deflections are converted to masses by the use of calibrating weights. The apparatus is built up from a "Meccano" set.

DISCUSSION

Mr. C. C. Paterson thought the instrument would have many useful applications. Had the author considered the possibility of making it a null method ? A simple electromagnetic arrangement could be employed for the purpose.

Prof. Bragg suggested a simpler null method, which consisted in the addition of weights to attachments to the side pieces. The arrangement could be such that quite considerable weights had to be added to produce a slight disturbance of the balance.

Mr. T. Smith asked if the author had tried reading the position of the fibre against a thick thread mounted parallel to it, instead of the inclined thread he had described. If the thick thread were brightly illuminated, and the fibre adjusted so that it appeared to bisect the width of the thread a very accurate setting would result.

Dr. D. Owen asked for the sensibility of the instrument shown at the meeting.

Mr. F. J. Harlow asked the form of the calibration curve.

Prof. Lees said the instrument reminded him of a well-known text-book method for measuring small horizontal forces, in which the force produced a small horizontal displacement of a weight suspended by a long thread. If two such suspended weights were connected by a horizontal fibre to which small loads could be attached, a micro-balance somewhat similar to Mr. Shaxby's would be obtained.

The Author, in reply, said Mr. Paterson's suggestion was interesting. He had made some attempts to convert the instrument into a null one with a magnetic control below the load. He had also tried Prof. Bragg's suggestion, by mounting weights on short horizontal arms near the fulcrums of the vertical rods. As regards sensitivity, readings could be repeated to about 1/500th of the maximum load for which the balance was adjusted. The calibration curve was nearly a straight line.

III. *The Resolution of a Curve into a Number of Exponential Components.*

By JOHN W. T. WALSH, *M.A.*, *M.Sc.* (*From the National Physical Laboratory.*)

RECEIVED SEPTEMBER 9, 1919.

Synopsis.—In many problems connected with the growth and decay of radio-active substances it is necessary to resolve an observed curve into two or more component exponentials of different half-periods.

Thus, for example, in a series of n successive transformations the amount of the nth product $N(t)$ at any time t is given by

$$N(t)=c_1e^{-\lambda_1 t}+c_2e^{-\lambda_2 t}+\ldots\ldots+c_ne^{-\lambda_n t}(*).$$

The problem dealt with in this Paper is the determination of the $2n$ constants $c_1\,c_2\,\ldots\,c_n$ and $\lambda_1\,\lambda_2\,\ldots\,\lambda_n$ from the form of the observed curve for $N(t)$.

Let it be assumed that the expression for the curve to be resolved has the form

$$B=a_1e^{-\lambda_1 t}+a_2e^{-\lambda_2 t}+\ldots+a_ne^{-\lambda_n t},$$

and let $B_0\,B_1\,B_2\,\ldots\,B_{2n-1}$ be $2n$ values of B at equidistant intervals of t. Then, assuming $a_1\,a_2\,\ldots$ to contain the constants $e^{-\lambda_1\tau}\,e^{-\lambda_2\tau}\,\ldots$ where τ is the starting point of the time series, and writing $\alpha_1\alpha_2\,\ldots$ for $e^{-\lambda_1 t}, e^{-\lambda_2 t}\,\ldots$

$$B_0=\Sigma a_1,$$
$$B_1=\Sigma a_1\alpha_1,$$
$$B_2=\Sigma a_1\alpha_1^2,$$
$$\ldots\ldots$$
$$B_{2n-1}=\Sigma a_1\alpha_1^{2n-1}.$$

If now $\alpha_1\alpha_2\,\ldots\,\alpha_n$ be the n roots of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0,$$

then

$$B_{(n+m)}+p_1B_{(n+m-1)}+\ldots+p_nB_m=0,$$

where $m<n$, for the left-hand member of this equation

$$=\Sigma a_1\alpha_1^{n+m}+p_1\Sigma a_1\alpha_1^{n+m-1}+\ldots+p_n\Sigma a_1\alpha_1^{m},$$
$$=\Sigma a_1[\alpha_1^{n+m}+p_1\alpha_1^{n+m-1}\quad+\ldots+p_n\alpha_1^{m}],$$

which is zero, for since $\alpha_1\alpha_2\,\ldots\,\alpha_n$ are the roots of the equation

* Rutherford, "Radioactive Substances and Their Radiations." 1913, p. 422.

in x written above, each term written in brackets vanishes separately. Hence we have n equations of the type

$$B_{n+m}+p_1B_{n+m-1}+\ .\ .\ .\ +p_nB_m=0,$$

where m has the n integral values 0 to $(n-1)$.

Hence, eliminating $p_1p_2\ldots p_n$ between these equations and the equation in x written above, we have

$$\begin{vmatrix} x^n & x^{n-1} & x^{n-2}\ldots & x & 1 \\ B_n & B_{n-1} & B_{n-2}\ldots & B_1 & B_0 \\ B_{n+1} & B_n & B_{n-1}\ldots & B_2 & B_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ B_{2n-1} & B_{2n-2} & B_{2n-3}\ldots & B_n & B_{n-1} \end{vmatrix}=0.$$

From this equation the values of $\alpha_1\alpha_2\ldots\alpha_n$ are found (graphically if necessary).

Substituting in the original equations the values of $\alpha_1\alpha_2\ldots$ thus obtained, the values of $a_1a_2\ldots$ are determined by means of any convenient set of n of the original equations for $B_0B_1\ldots.$

As an example of analysis into two exponentials let us take a case in which the observed values at 100, 400, 700 and 1,000 days were 278, 107, 70 and 50 units respectively.

Then

$$B_0=278$$
$$B_1=107$$
$$B_2=\ 70$$
$$B_3=\ 50$$

and

$$\begin{vmatrix} x^2 & x & 1 \\ 70 & 107 & 278 \\ 50 & 70 & 107 \end{vmatrix}=0$$

i. e., $$8{,}011x^2-6{,}410x+450=0,$$

giving $$x=0{\cdot}7014 \quad\text{and}\quad 0{\cdot}0769.$$

Hence $$a+b=278$$

and $$0{\cdot}7014a+0{\cdot}0769b=107$$
$$0{\cdot}6245a=107-21{\cdot}4=85{\cdot}6,$$

$\therefore$ $$a=137{\cdot}1,$$

and $$b=140{\cdot}9.$$

Hence

$$B=137{\cdot}1\times(0{\cdot}7014)^{\frac{t-100}{300}}+140{\cdot}9\times(0{\cdot}0769)^{\frac{t-100}{300}}$$
$$=154{\cdot}3\ e^{-0{\cdot}00118t}+331{\cdot}4\ e^{-0{\cdot}00855t},$$

where t is expressed in days, from the point $t=0$.

If, as is usual, the observations are taken over an extended period, possibly at irregular intervals of time, the values of $B_0 B_1 \ldots$ required by the above method are obtained by plotting the observed values, drawing a curve through them by eye, and then reading from the curve the values at the time interval selected.

It is, however, desirable to ensure that the values of the constants found represent the *whole* of the observed values so as to make the probable error a minimum. This may be done by finding, from the calculated expression, the value of the function at each observed point of the curve. If ∂_r be the difference between the calculated and the observed values at any point, and if $a_1 \ldots a_n$ and $\lambda_1 \ldots \lambda_n$ be the values of a and λ found as above, then if the most probable values of these quantities be $a_1(1+\theta_1) \ldots$ and $\lambda_1(1+\phi_1) \ldots$, θ and ϕ can be found from all the available equations of the type

$$\partial_r = \Sigma a_1(1+\theta_1)e^{-\lambda_1(1+\phi_1)t} - \Sigma a_1 e^{-\lambda_1 t}.$$

Now if θ and ϕ are small, so that squares and products may be neglected, this becomes

$$\partial_r = \Sigma \{a_1 e^{-\lambda_1 t}\theta_1 - a_1\lambda_1 t\phi_1\},$$

and from all the available observations, the most probable values of $\theta_1 \ldots$ and $\phi_1 \ldots$ may be found by forming the normal equations in these quantities as usual.

As an example, seven of the observations from which the curve was drawn to give the values used in the example on p. 27 were as follows :—

TABLE I.

t.	B_t.	∂r.	B_t (calc.).	∂_r'.
41	402	+21·3	398·7	+3·3
74	325	+ 7·5	323·6	+1·4
139	224	− 7·8	227·3	−4·9
295	126	− 9·5	129·8	−3·8
443	99	+ 0·1	96·3	+2·7
669	71	− 0·3	70·3	+0·7
1,205	42	+ 4·6	36·9	+5·1

Using the expression found for B_t on p. 27, we can obtain the values of ∂_r given in the third column above. From these values the following normal equations result :—

$$85266\ \theta_1 + 76129\ \theta_2 + 19960\ \phi_1 + 53905\ \phi_2 + 2295 = 0$$
$$76129\ \theta_1 + 96588\ \theta_2 + 7444\ \phi_1 + 52912\ \phi_2 + 5260 = 0$$
$$19960\ \theta_1 + 7444\ \theta_2 + 10253\ \phi_1 + 8871\ \phi_2 - 51{\cdot}9 = 0$$
$$53905\ \theta_1 + 52912\ \theta_2 + 8871\ \phi_1 + 38912\ \phi_2 + 1015 = 0$$

These give

$$\theta_1 = +0{\cdot}0039$$
$$\theta_2 = +0{\cdot}134$$
$$\varphi_1 = +0{\cdot}0063$$
$$\varphi_2 = +0{\cdot}148_5$$

Hence $$B = 154{\cdot}9\, e^{-0{\cdot}00119t} + 375{\cdot}8\, e^{-0{\cdot}00982t}.$$

The values of B_t found from this expression, and the difference from the observed values ∂_r', are shown in columns 4 and 5 respectively of Table I.

The method is, of course, analogous to the Lagrange-Dale method of analysis of a compound periodic function,* simplified for the less complicated case of pure exponentials.

It is clear that a similar method may be employed to analyse a curve of the form $Q = \Sigma a_1 e^{-\lambda_1 t} \cos(p_1 t + \alpha_1)$, for this is identical with the form

$$Q = \Sigma \{A_1 e^{-l_1 t} + B_1 e^{-m_1 t}\}.$$

Appendix.

The following method of computing the determinants of the fourth order required for the treatment of a compound exponential of two elements may be of interest. If a determinant of the fourth order be

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

it may be written as

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \times \begin{vmatrix} c_3 & c_4 \\ d_3 & d_4 \end{vmatrix} + \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix} \times \begin{vmatrix} c_1 & c_2 \\ d_1 & d_2 \end{vmatrix} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \times \begin{vmatrix} c_2 & c_4 \\ d_2 & d_4 \end{vmatrix}$$
$$- \begin{vmatrix} a_2 & a_4 \\ b_2 & b_4 \end{vmatrix} \times \begin{vmatrix} c_1 & c_3 \\ d_1 & d_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} \times \begin{vmatrix} c_2 & c_3 \\ d_2 & d_3 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \times \begin{vmatrix} c_1 & c_4 \\ d_1 & d_4 \end{vmatrix}$$

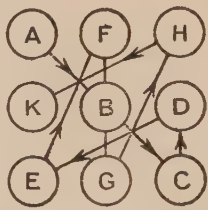
Thus the 5 fourth order determinants in the solution of the four normal equations may be expressed as sums of products of 20 minors of the second order.

The sixth order determinant may be similarly resolved either into second or third order minors. If third order minors

* J. B. Dale, Roy. Astron. Soc., M.N., 74, p. 628, May, 1914.

are used, work with logarithms may be facilitated by the following scheme of writing :—

Write $\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & K \end{vmatrix}$ as



where (A) stands for log A, etc.

Adding each row and column, and writing down the anti-logarithms, the difference between the sum of the anti-logarithms of the rows and the sum of the anti-logarithms of the columns is the value of the determinant, for this

$$=AEK+BFG+CDH$$
$$-GEC-HFA-KDB.$$

ABSTRACT.

The Paper gives a method for the resolution of a curve of the compound exponential form $B=\sum_1^n a_1e^{-\lambda_1 t}$ into its components, the values of a and λ for the n different exponential terms being found from $2n$ values of B equidistant along the axis of t. A method is also given for finding the most probable values of these constants from any number ($>2n$) of observed values of B taken at irregular intervals of t.

DISCUSSION.

Mr. T. SMITH said the crux of the matter appeared to lie in the proper choice of the number of components, and this was not touched on in the Paper. It was possible to come to a decision provided the observations covered a sufficiently long period. Suppose one felt justified in taking n terms, but was uncertain whether or not to go to $n+1$, a comparison of the outstanding errors when first n and then $n+1$ terms were taken wo uld reveal whether there was any advantage in introducing the extra one.

Prof. BRAGG asked whether in the event of uncertainty as to taking n or $n+1$ terms it made much difference which was adopted. Did all the old λ's change and another one of comparable magnitude come on?

Mr. T. SMITH said the new λ might well be quite large.

Mr. E. H. RAYNER pointed out that it depended on the relative values of a, and λ whether there would be much difference in taking n or $n+1$.

Prof. LEES asked if the method worked when the a's were negative ?

The AUTHOR, in reply, said that in some curves the value of n was known from a priori considerations ; and in some other cases it was sufficient to take enough terms to fit the experimental curve well. The modifications of the λ's in taking $n+1$ instead of n terms was not simple. In ordinary cases all the λ's would be varied, and a new one comparable with the others added. The method applied equally to negative a's.

IV. *On the Self-inductance of Single-layer Flat Coils.* *By* S. BUTTERWORTH, *M.Sc., The National Physical Laboratory.*

RECEIVED OCTOBER 1, 1919.

1. The extended use of flat inductance coils in high-frequency electrical measurements renders desirable the development of simple formulæ for the predetermination of the inductance of such coils. A modification of Weinstein's formula* for coils of rectangular channel section has usually been employed for this purpose. This formula fails, however, when the inner radius is small.

It is the purpose of the present Paper to extend the modified Weinstein formula, to supply a formula suitable for coils of small inner radius, and to use the two formulæ to compute a table (Table I.) for the use of designers of flat inductance coils. Some applications of this table are also given.

2. The method of obtaining the formulæ is one of integration, starting with a suitable formula for the mutual inductance between two coaxial co-planar circles.

Maxwell† has given two elliptic integral formulæ for the mutual inductance between two coaxial circles. When the circles are also co-planar these reduce to

$$M=4\pi\sqrt{Aa}\,\{(2/k-k)K(k)-2/kE(k)\}, \quad . \quad . \quad . \quad . \quad (1)$$

or,
$$M=8\pi A\,\{K(a/A)-E(a/A)\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

in which K and E are complete elliptic integrals of the first and second kind respectively; a, A are the radii of the circles, and
$$k^2=4Aa/(A+a)^2.$$
In (2) $A>a$ always.

3. *Derivation of Extended Weinstein Formula.*

Let R be the mean radius of the flat coil, $2X$ be the coil depth, and n be the number of turns per unit of length. Also let $m(x, r)$ be the mutual inductance between two co-planar coaxial circles, whose mean radius is r and radius difference is $2x$. Then by summation of the mutual inductance between

* Weinstein, "Wied. Ann.," XXI., p. 329, 1884; Bull. Bureau of Standards, VIII., p. 137, 1912.

† "Electricity and Magnetism," Vol. II., 701.

the elementary filaments of the flat coil the self-inductance of the latter is

$$L=4n^2\int_0^X dx\int_{R-X+x}^{R+X-x} m(x,r)dr. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)^*$$

The integral (3) may be evaluated by expressing m in terms of x and r. The required expression may be derived from (1) by developing into a series involving only k' the complementary modulus of k, for

$$k'=(1-k^2)^{\frac{1}{2}}=(A-a)/(A+a)=x/r.$$

The suitable series† is

$$M=4\pi r\left(\Phi_0+\frac{1^2}{2^2}k'^2\Phi_1+\frac{1^2\,.\,1^2}{2^2\,.\,4^2}k'^4\Phi_2+\frac{1^2\,.\,1^2\,.\,3^2}{2^2\,.\,4^2\,.\,6^2}k'^6\Phi_3+\ldots\right) \quad (4)$$

in which

$$\left.\begin{aligned}\Phi_0&=\log_e 4/k'-2\\ \Phi_1-\Phi_0&=\frac{1}{1}+\frac{2}{1}\\ \Phi_2-\Phi_1&=\frac{1}{2}-\frac{2}{1}\\ \Phi_3-\Phi_2&=\frac{1}{3}-\frac{2}{3}\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ \Phi_n-\Phi_{n-1}&=\frac{1}{n}-\frac{2}{2n-3}\end{aligned}\right\}.$$

Applying (3) to the first four terms of (4)

$$L=16\pi n^2RX\left\{(\lambda-\tfrac{1}{2})+\tfrac{2}{3}z^2\left(\lambda+\frac{43}{12}\right)+\frac{44}{45}z^4\left(\lambda+\frac{96}{55}\right)\right.$$
$$\left.+\frac{412}{105}z^6\left(\lambda+\frac{98579}{86520}\right)+\ldots\right\} \quad (A)$$

in which

$$\lambda=\log_e 4R/X,\ z=X/4R.$$

The first two terms of this series constitute the modified Weinstein formula.

The series converges for all possible values of X and R.

* For the method of building up this integral, *see* Butterworth, "Proc." Phys. Soc., XXXVII., p. 376, 1915.

† Bromwich, "Quarterly Journal of Pure and Applied Mathematics," No. 176, p. 363, 1913; Butterworth, "Phil. Mag.," XXXI., p. 276, 1916.

In the worst case when the inner radius is zero $X/R=1$, and the value of the terms in { } of (A) are then

0·886 0·207 0·012 0·002.

Thus, assuming the four terms evaluated approximate sufficiently closely to the true sum, the Weinstein formula gives a result about 1 per cent. low in the extreme case.

To verify this result a formula is now developed which is suitable for large values of X, that is, for a small inner radius of the flat coil.

4. Using Maxwell's second formula {(2) above} and putting G for $K-E$, the flux Φ through a circle of radius x in the plane of the coil is

$$8\pi n\left\{x\int_{r_1}^{x} G(a/x)da+\int_{x}^{r_2} aG(x/a)da\right\},$$

in which r_1, r_2 are the inner and outer radii, or by a change of variables

$$\Phi=8\pi nx^2\left(\int_{r_1/x}^{1} G(\mu)d\mu+\int_{x/r_2}^{1} G(\mu)d\mu/\mu^3\right). \quad . \quad . \quad . \quad (5)$$

The self-inductance of the flat coil is

$$L=n\int_{r_1}^{r_2}\Phi dx,$$

so that, integrating (5) by parts

$$L=\frac{16}{3}\pi n^2r_2{}^3\int_{a}^{1}(1-a^3/\mu^3)G(\mu)d\mu, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which $$a=r_1/r_2.$$

The evaluation of (6) involves integration of K and E with respect to the modulus. The required integrations have been discussed by the author in an earlier Paper.*

Using the notation and results obtained therein, (6) reduces to

$$L=\frac{8}{3}\pi n^2r_2{}^3\{u_1-1-a^3(1+v)-u+2aE-a(1-a^2)K\}, \quad . \quad . \quad (7)$$

in which

$$u=\int_0^a Kd\mu,\ v=\int_a^1 Kd\mu/\mu,$$

$$u_1=\int_0^1 Kd\mu=1\cdot831931248 \quad . \quad . \quad .$$

* "Phil. Mag.," XXIX., p. 584, 1915.

(7) may be expressed in series by making use of the series for K, E, u, v, giving

$$L=\frac{4}{3}\pi^2 n^2 r_2^3[2(u_1-1)/\pi+a^3\{2(u_1-1)/\pi+\tfrac{1}{6}-\log 4/a+\sigma\}] \quad (B)$$

in which

$$\sigma=6\sum_1^{\infty}\left(\frac{1\,.\,3\,.\,5\,\ldots\,2n-1}{2\,.\,4\,.\,6\,\ldots\ldots\,2n}\right)^2\frac{2n+1}{2n(2n+1)(2n+3)}a2n.$$

This formula is most convergent when a is small and is thus suitable for coils having a small inner radius.

5. For the case of zero inner radius formula (B) gives

$$L\doteq 6{\cdot}96957n^2r_2^3,$$

formula (A) gives

$$L=6{\cdot}956n^2r_2^3,$$

while the first two terms only give

$$L=6{\cdot}87n^2r_2^3.$$

Thus, in the most unfavourable case for formula (A) the error is of the order 0·2 per cent., while the Weinstein formula gives a result 1·5 per cent. below the true value. For the case $a=0{\cdot}5$

$$L=4{\cdot}743502n^2r_2^3 \text{ by } (A),$$

$$L=4{\cdot}743500n^2r_2^3 \text{ by } (B).$$

The two formula thus check each other.

For general computation formula (A) usually the most rapid to work with, while formula (B) may be used as a check formula.

6. For practical calculations we may write

$$L=Qn^2r_2^3, \quad \ldots\ldots\ldots\ldots \quad (C)$$

and tabulate Q as a function of r_1/r_2. This is done in Table I.

TABLE I.

r_1/r_2.	Q.	r_1/r_2.	Q.	r_1/r_2.	Q.
0·00	6·970	0·35	5·996	0·70	2·528
0·05	6·964	0·40	5·632	0·75	1·946
0·10	6·930	0·45	5·213	0·80	1·397
0·15	6·845	0·50	4·743	0·85	0·8892
0·20	6·728	0·55	4·231	0·90	0·4574
0·25	6·544	0·60	3·682	0·95	0·1394
0·30	6·300	0·65	3·105	1·00	0·0000

Self-inductance in centimetres $=Qn^2r_2^3$.
r_1=inner radius (centimetres). n=turns per centimetre.
r_2=outer radius (centimetres).

7. *Flat Coil of Best Time Constant.*

The (direct current) resistance of a flat coil wound closely with circular wire is

$$4\rho n^3(r_2^2-r_1^2)=4\rho n^3 r_2^2(1-\alpha^2),$$

ρ being the resistivity of the wire, so that the time constant is proportional to $Qr_2/n(1-\alpha^2)$.

Also for a given length and section of wire $r_2^2(1-\alpha^2)$ and n are fixed, and therefore (by eliminating r_2) the time constant is proportional to $Q/(1-\alpha^2)^{3/2}$. By calculating this quantity for various radius ratios we obtain the variation of time-constant. It is found (*see* Table II.) that the maximum time constant occurs when the inner radius is about 2/5 the outer radius. The maximum is, however, very flat, the coil having 90 per cent. of the maximum efficiency if $r_1/r_2<0{\cdot}7$.

TABLE II.

$\alpha=$	0·0	0·1	0·2	0·3	0·4	0·5
$Q/(1-\alpha^2)^{3/2}=$	6·97	7·03	7·15	7·25	7·32	7·30
$\alpha=$	0·6	0·7	0·8	0·9	1·0	
$Q/(1-\alpha^2)^{3/2}=$	7·18	6·94	6·46	5·50	0·00	

The best time constant for a single layer *cylindrical* coil is obtained when the length (b) is 4/5 of the radius (a), and in that case $L=14{\cdot}90n^2a^3$, n again being the number of times per centimetre. For a flat coil of the best time constant

$$L'=5{\cdot}632n^2r_2^3.$$

To compare the two cases we require the radius a of the cylindrical coil which can be wound with the same length of wire as the flat coil. This is $a=0{\cdot}725r_2$, with $r_1=0{\cdot}4r_2$.

Using this $$L=5{\cdot}666n^2r_2^3.$$

The cylindrical coil is thus slightly better than the corresponding flat coil.

8. *Mutual Inductance between Coaxial Coplanar Flat Coils.*

The mutual inductance between two coaxial flat coils in the same plane can be obtained from Table I. as follows :—

Let the inner and outer radii of the $\left\{\dfrac{\text{inner}}{\text{outer}}\right\}$ coils be

$$\left\{\frac{r_1,\ r_2}{r_3,\ r_4}\right\}.$$

Denote the self-inductances of the two coils by L_α, L_γ, and that of the coil which would fill the interspace by L_β.

Denote the mutual inductances between these three coils by $M_{\alpha\beta}$, $M_{\beta\gamma}$, $M_{\gamma\alpha}$. Further, let L_α, L_β in series have self-inductance L_A; L_β, L_γ in series have inductance L_B; L_α, L_β, L_γ in series have inductance L_C.

Then $L_A=L_\alpha+L_\beta+2M_{\alpha\beta}$, $L_B=L_\beta+L_\gamma+2M_{\beta\gamma}$, $L_C=L_\alpha+L_\beta+L_\gamma+2M_{\alpha\beta}+2M_{\beta\gamma}+2M_{\gamma\alpha}$, from which the three M's can be found since the L's are known from Table I.

In particular

$$M_{\alpha\gamma}=\tfrac{1}{2}(L_C-L_A-L_B-L_\beta),$$

or, in terms of Q

$$M_{\alpha\gamma}=\tfrac{1}{2}n^2\{r_4^3(Q_{r_1/r_4}-Q_{r_2/r_4})-r_2^3(Q_{r_1/r_3}+Q_{r_2/r_3})\}.$$

A simple case is where there is no interspace, and the inner coil has zero inner radius. Then

$$M=\tfrac{1}{2}n^2R^3\{Q_0(1-\alpha^3)-Q_\alpha\},$$

in which $\alpha=r/R$, R is the outer radius, r the dividing radius.

In order to give some idea of the magnitude of the various inductances, the following table has been calculated from the above formula :—

TABLE III.—*Mutual-inductances Between Flat Coils.*

α.	L_1/n^2R^3.	L_2/n^2R^3.	M/n^2R^3.	k.
0·1	0·00697	6·93	0·0162	0·074
0·2	0·0557	6·73	0·0930	0·152
0·3	0·1880	6·30	0·240	0·220
0·4	0·446	5·63	0·446	0·280
0·5	0·871	4·74	0·678	0·333
0·6	1·503	3·68	0·892	0·379
0·7	2·39	2·53	1·025	0·421
0·8	3·56	1·397	1·005	0·451
0·9	5·08	0·457	0·715	0·470
0·92	5·23	0·317	0·611	0·474
0·94	5·79	0·1910	0·495	0·470
0·96	6·17	0·0942	0·356	0·465
0·98	6·56	0·0272	0·192	0·454
1·00	6·97	0·0000	0·000	0·000

L_1=Self-inductance of inner coil.
L_2=Self-inductance of outer coil.
M=Mutual inductance between coils.
k=Coefficient of coupling$=M/\sqrt{L_1L_2}$.
r=Dividing radius.
$\alpha=r/R$. n=Turns per centimetre.

ABSTRACT.

Two formulæ are established for the computation of the self-inductance of single layer flat coils, one for the case when the inner and outer radii are not very different and the other for the case of small inner radius. The two formulæ are shown to be consistent and capable

of including all possible cases. From the formula a table is calculated which enables the inductance to be expressed in the form $L=Qn^2r^3$, in which n is the number of turns per cm., r the outer radius and Q is a tabulated function of the ratio of the inner and outer radii. Some applications of the table are given.

DISCUSSION.

Dr. Eccles recalled the very useful Paper published by the author some years ago, giving formulæ for thick coils. He had found these extremely useful, and felt sure that the results of the present Paper would prove equally valuable. He felt that people hardly realised their indebtedness to those who undertook these very laborious calculations for the help of the users of coils.

Dr. E. H. Rayner pointed out how useful such formulæ were in the construction of inductances. Almost anyone could make up a resistance to have any desired value; but it was another matter with regard to inductances. With formulæ such as these, however, it was possible with the simplest appliances to construct an inductance to within 1 per cent. of the required value.

Prof. G. W. O. Howe communicated the following: In the "Archiv für Elektroteknik," Vol. 3, 1915, page 187, is a Paper by J. Spielrein, entitled "The Self-Induction of Air-Core Spirally-wound Coils" (*see* "Science Abstracts," Elec. Eng., 1915, No. 660). Spielrein's final formula is $L=T^2r_2f(r_1/r_2)$, where T is the total number of turns, whereas Mr. Butterworth's formula is $L=n^2r^3Q$, where n is the turns per cm. Since $T=n(r_2-r_1)$ Spielrein's formula can be written $L=n^2(r_2-r_1)^2r_2f(r_1/r_2)$. Hence $Q=(r_2-r_1/r_2)^2f(r_1/r_2)=(1-r_1/r_2)^2f(r_1/r_2)$.

Now Spielrein gives a table of $f(r_1/r_2)$ to seven places for all values of r_1/r_2 from 0 to 1 by steps of 0·01. Comparing Spielrein's $f(r_1/r_2)$ with Mr. Butterworth's Q we have :—

r_1/r_0.	$f(r_1/r_2)$.	$(1-r_1/r_2)^2f(r_1/r_2)$.	Q.
0·0	6·969573	6·969573	6·970
0·5	18·97400	4·74350	4·743
0·9	45·74241	0·4574241	0·4574
1·0	∞	$0\times\infty$	0

It seems obvious that Mr. Butterworth was unaware of this Paper.

The Author, in reply, said he had not known that Spielrein had tackled the problem. He was afraid it rendered his Paper somewhat unnecessary.

Dr. Eccles pointed out that it was of great importance that two independent investigators obtained results in such striking agreement.

V. *An Experimental Method of Determining the Primary Current at Break in a Magneto.* By N. W. McLachlan, *D.Sc., M.I.E.E.* (*From the National Physical Laboratory.*)

(Communicated by F. E. Smith, O.B.E., F.R.S.)

Received October 3, 1919.

In carrying out research on various types of magneto, it is sometimes necessary to know the value of the current broken in the primary circuit. Since the effective resistance and inductance of the primary winding under conditions prior to break are small, it is difficult to ensure that the insertion of apparatus in the primary circuit will not alter the usual working conditions. For this reason the oscillograph is out of the question for accurate measurement. Mr. A. P. Young* has devised graphical and analytical methods of determining the shape and magnitude of the primary current up to the point of break. With a view to facilitating the measurement of the primary current at break the following method has been developed.

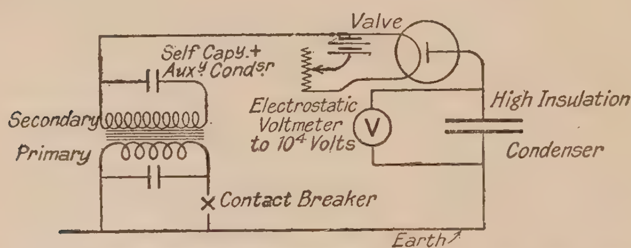


Fig. 1.—Diagram showing Arrangement of Apparatus for Reducing and Measuring the Peak Voltage of a Magneto.

Apart from the voltage after break due to rotation, the secondary peak potential of a magneto is very nearly proportional to the current broken in the primary circuit. Thus in order to determine the current broken at various speeds, the peak voltage is found and a curve plotted showing the relation between voltage and speed for any definite position of the timing lever. This curve must be corrected, the ordinates being reduced by an amount equal to the voltages due to rotation at the speeds in question. The apparatus used is shown in Fig. 1†. The voltage rises with the speed

* "Journal" Aeronautical Society, No. 82, April, 1917.
† Campbell and Paterson, "Phil. Mag.," Vol. XXXVII., March, 1919.

until in the neighbourhood of 100 revs. per min., sparks occur at the safety gap. In order to reduce the voltage of the magneto below that necessary to spark the safety gap, either a definite non-inductive leak can be allowed in the secondary circuit, or a condenser can be connected across the secondary terminals, or from the high tension lead of the secondary to earth. The condenser is probably preferable, since for a given reduction in peak voltage it will not affect the primary current before break as much as a resistance. The capacity (and the power factor) of the condenser should be as small as possible, and it should not leak. The capacity used in these experiments, including the distributed capacity of the secondary winding, was 600 picofarads. In this way a curve showing the relation between secondary peak voltage and speed can be obtained. In order to determine the voltage after break due to rotation,

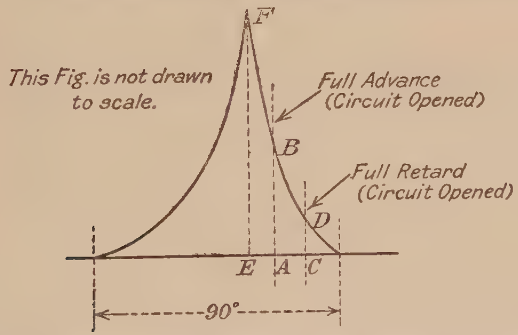


Fig. 2.—Diagram showing Primary Open Circuit Voltage Wave of Magneto.

the magneto is run at various speeds with the condenser on the secondary, the contacts of the breaker being insulated, *i.e.*, on open circuit. The relation between voltage and speed with the condenser was not quite linear, owing to the increased condenser effect at high speeds. The effect of the condenser was to increase the voltage due to rotation by 15 per cent. at 1,200 revs. per min. The voltage determined in this case was the rotational peak voltage. To estimate the fraction of the latter voltage, which must be subtracted from the total peak voltage in order to obtain that due solely to interruption of the current, the open circuit voltage curve is obtained by means of an oscillograph, or a point to point method. In using the latter method, it is necessary to find the rotational open circuit voltages at advance and retard

only. When the open circuit voltage curve is used, a vertical line is drawn at either the full advance or full retard positions, as shown in Fig. 2. This curve is taken from Mr. Young's paper (*loc. cit.*), and may be adopted for the magneto used in these experiments without introducing appreciable errors.* The open circuit peak voltage due to rotation is EF and the voltage at advance break is AB and at retard DC. Knowing the ratios AB/EF, DC/EF, and the value of EF as found from a speed calibration test, the value of the voltage to be subtracted from the total peak voltage can be calculated. Allowance can be made for the lag of the peak voltage behind the breaking of the current, if necessary, by shifting AB and CD

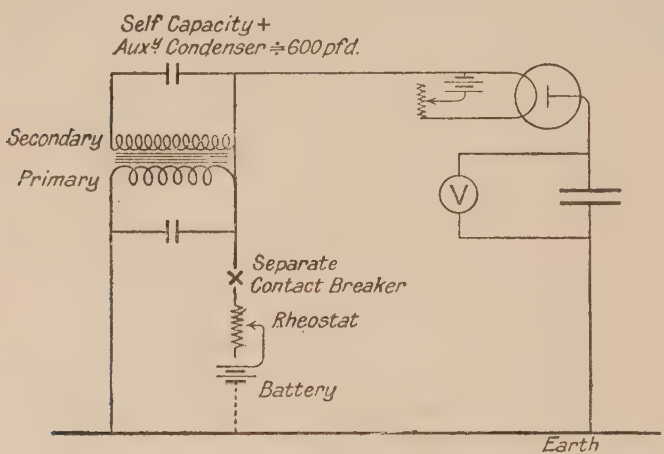


Fig. 3.—Arrangement of Circuit used for Calibrating Magneto.

to the right by the requisite amount, but this refinement is probably superfluous.

The system is then calibrated by using the circuit shown in Fig. 3.

A battery and non-inductive rheostat are connected in circuit with a separate contact breaker and the primary of the magneto. In order to reduce earth capacity effects (*see* Fig. 6), and the effect of transients through the battery and

* On full advance the correction is about 15 per cent. when the speed is 2,000 revs. per min., so that an error of 10 per cent. in the voltage due to rotation is only an error of 1·5 per cent. in the primary current at break. On retard the correction is 2·5 per cent. at 2,000 revs. per min. The correction decreases with decrease in speed, and is very small on the initial portions of the curves of Fig. 4.

rheostat before and after break to a minimum, the connections of Fig. 3 should be adopted. In this respect it is advisable to use a portable battery in preference to a portion of a large main battery. The armature of the magnets is clamped in the position at which break would occur during rotation, *e.g.*, full advance or full retard. Precautions are taken to ensure a sufficiently small time constant, so that the current through the armature attains its maximum value before the circuit is broken. The current should produce a magnetic field *in the same direction* as that of the induced current when the magneto is operated in the usual manner. In order to allow for leakage from the peak voltage measuring apparatus, the contact breaker should run at about 600 revs. per min. With two breaks per revolution this gives 20 breaks per second. A series of readings of current broken in the primary of the magneto and the peak voltage are taken.

There is a slight difference between the working conditions and those during calibration. Under running conditions, using a condenser across the secondary, the current through the armature and the flux are reversed, whereas with a fixed rotor and a separate contact breaker the flux is never reversed. The induced voltage wave of the magneto is peaked, whereas with the contact breaker and a battery the voltage is constant. Moreover, the flux change in the iron before break is different in the two cases. Thus the magnetic history of the iron will be different. These variations in conditions, however, ought not to have a very appreciable effect, and it is probable that the results are sufficiently accurate for all practical requirements. Whatever may be the difference between the two conditions, the method yields results which are useful for purposes of comparison.

In order to ascertain the effect of the condenser across the secondary on the value of the current broken in the primary, found by the method already described, comparative tests were carried out with and without the condenser. The voltmeter employed would not read above 10,000 volts, owing to the nearness of the fixed and moveable vanes, so that readings could not be taken at a speed of more than 100 evs. per min. This difficulty was overcome by using either a core of thick iron, *e.g.*, strips about 4 mm. thick, or a core of the same type as that used in practice, but of half the cross-sectional area. The actual core was also used, the voltage being reduced by introducing air gaps into the magnetic

circuit between the laminated vertical limbs in the housing and the core. The peak voltage was found at various speeds up to 2,000 revs. per min. with and without the condenser for different positions of the timing lever. The ratio of the peak voltages, with and without the condenser at full retard, was about 3 per cent. less at 80 revs. per min. than at 2,000 revs. per min. On full advance there was a closer agreement at all speeds.

The ratio of the peak voltages found by direct current calibration, was about 2 per cent. less than that at 80 revs. per min. and 5 per cent. less than that at 2,000 revs. per min., when the timing lever was fully retarded. At advance the agreement was better. Thus, considering the difficulty of obtaining accurate results in magneto research, the effect of the condenser, although it increases slightly with the speed, is not of much importance. The increase in peak voltage is probably chiefly due to the effect of the condenser on the behaviour of the iron (it decreases the damping 27 per cent. owing probably to the reduction in the frequency of the oscillations), but there is doubtless a slight increase in the primary current at high speeds. It was not always possible to repeat the readings to within 1 per cent. Variations of 3 per cent. were obtained and it was observed that these were connected with the previous history of the iron.

The current at break can also be found by using an armature having the same type of core and the same number of primary turns as that used with the condenser. The number of secondary turns must be reduced from 10,000 to 3,000, so that the peak voltage can be obtained at all speeds. The armature is calibrated in the manner indicated above. Using an armature of this type there was a close agreement between the values obtained by the two methods at full advance, but an increase of 20 per cent. at low speeds with the timing lever fully retarded. At high speeds the armature gave values about 2 per cent. low. These differences can probably be accounted for by inequality of the iron cores, and by the fact that the small armature was wound on a brass former whilst the other was not. In the advance and retard positions the configuration of the magnetic circuit and the polarisation due to the magnet are different, owing to the altered position of the rotor.* The value of dB/dt is different, and the mag-

* The rotor was partially laminated, and the proportion of the oscillatory magnetic circuit which was laminated at retard, was greater than at advance.

nitude of the negative flux is greater in the latter case than in the former. The highest flux density in the armature is greater at advance than at retard, but the total flux change between make and break is greater at retard than at advance (*see* Fig. 9). It is for this reason that the current and energy at high speeds, with the timing lever fully retarded, are greater than the same quantities at full advance. At low speeds the current and energy at break are less on retard than advance, because in the one case break occurs after the current has attained its maximum value, whereas in the other it occurs when the current is almost a maximum. This is due to the fact that at low speeds the effective resistance of the primary winding is of greater importance than the inductive reactance. Thus as the speed increases, the angle (on the time axis) between make and the occurrence of the maximum primary current also increases. Owing to eddy current and hysteresis losses the effective resistance of the primary increases with the speed. Since the screening effect of the eddy currents increases with the speed, the effective permeability of the iron decreases, and therefore the inductance of the primary in any given position decreases. Hence, there is a certain speed for which the current at break is a maximum.

It follows from the preceding that two apparently similar cores which gave approximately the same value of the current at advance would not necessarily do so at retard. Another armature having standard primary and secondary windings, viz., 150 turns of 22 S.W.G. and 10,000 of 41 S.W.G. enamelled wire, wound on a brass former, was tried. The readings were in close agreement at advance, but at retard they were 15 and 4 per cent. less at 90 revs. per min. and 1,700 revs. per min. respectively, than those obtained with the reduced armature. At low speeds on full retard or advance it was found that the primary current was the same with and without the condenser. It is evident, therefore, that the performance in these cases depends chiefly on the quality of the iron cores, although with a condenser across the secondary the transient oscillations which occur after make, may influence the magnetic behaviour of the iron. It is desirable in practice to use a condenser across the secondary in preference to an armature with a reduced secondary. In the former case the current under actual working conditions is obtained and assumptions have not to be made with regard to the quality of the iron cores.

The results of some experiments on an inductor type, B.T.H., A.V.8s magneto* are given in Fig. 4. It should be mentioned that the magnet was not magnetised to its maximum

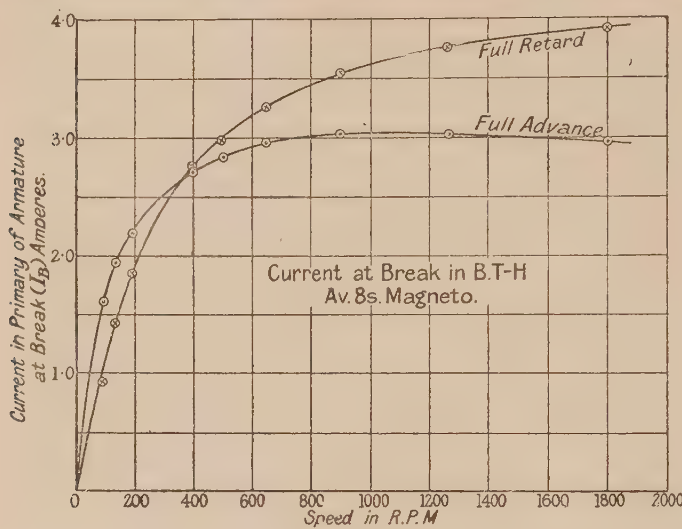


Fig. 4.

extent, since the machine had been dismantled before the tests were carried out. This, however, only affects the absolute magnitude of the current in the armature.

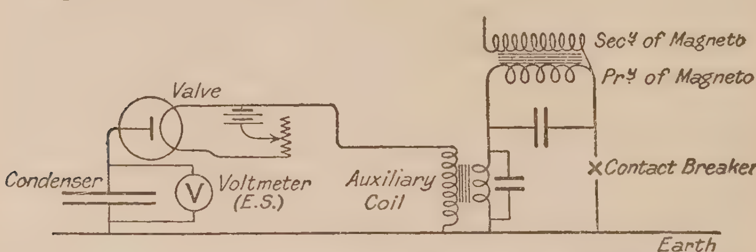


Fig. 5.—Diagram of Apparatus for Measuring the Primary Current at Break in a Magneto by the aid of an Auxiliary Induction Coil.

It may be of interest to outline another method of measurement which was tried, but without success. The circuit used is illustrated in Fig. 5.

* The experiments described herein refer to this type of magneto, unless otherwise stated.

An auxiliary iron-cored induction coil with a primary of very small inductance (5 microhenries at 50 ∿) and resistance 0·005 ohm, is inserted between one end of the primary winding of the magneto and earth (not a virtual but an actual earth). There is no mutual inductance between the coil and the magneto. The auxiliary coil, as shown in Fig. 5, does not appear to be in the primary or secondary oscillatory circuits after break, provided the magneto is *prevented from sparking*, and provided the high-frequency oscillations in the primary do not pass through the initial arc or spark when the contacts of the breaker separate.

A condenser of about 0·1 microfarad is connected across the terminals of the auxiliary coil to prevent sparking at the contact breaker, since the magneto condenser is not connected directly across the breaker. The secondary winding of the auxiliary coil is connected to earth and to the filament of a

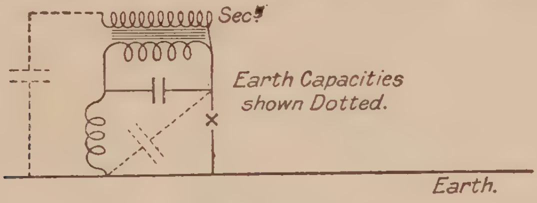


Fig. 6

two-electrode ionic valve which serves to rectify the oscillations. The anode of the valve is connected to an electrostatic voltmeter reading to 160 volts with a condenser in parallel to allow for voltmeter leakage, which latter must be extremely small. The other terminal of the voltmeter is connected to earth. It would appear that the reading of the voltmeter is a measure of the current broken in the primary of the auxiliary coil, and therefore of the current in the primary of the magneto. This, however, was found not to be the case as will be explained later.

It is well known that sparking or flashing occurs periodically at the contact breaker of a magneto. It was found that the voltmeter was subjected to violent impulses from time to time, the intervals between the impulses decreasing with increase in the speed of rotation. They were particularly noticeable during changes of speed and at high speeds. At low speeds their effect could be eventually wiped out by allowing a slight leak on the voltmeter, and by increasing the

primary capacity. This, however, caused a large error in the reading. It was suspected that these impulses were due to the oscillations in the primary of the magneto traversing the primary of the auxiliary coil via the arc at the contact breaker. There is also another possible reason for the passage of oscillatory current through the primary of the auxiliary coil. This is due to capacity effect to earth of the magneto windings. This can be represented by condensers as shown dotted in Fig. 6.

If the magneto is allowed to spark, high frequency oscillations pass through the auxiliary coil and increase the reading of the voltmeter enormously.

The auxiliary coil is calibrated using the circuit of Fig. 3. Owing to the voltmeter leakage being large to allow for impulse effects, the reading increased with the speed of rotation of the contact breaker. The method is, therefore, unreliable.

Investigation into the Effect of a Condenser, Across the Secondary Terminals of a Magneto on the Primary Current at Break.

The induced voltage wave of a magneto is peaked as shown in Fig. 2, and therefore contains pronounced higher harmonics. In the analytical treatment of the above problem we will for simplicity, since the condenser effect is only of the second order of magnitude, assume that the induced primary and secondary voltages are sinusoidal in wave form. It is the slope of the voltage wave which is the determining factor for the capacity current, but we can virtually increase the slope by taking an equivalent frequency greater than that of the alternation of the flux. Thus, at high speeds of, say, 1,800 revs. per min. the effect of capacity will be much greater than at a speed of 80 revs. per min.

The oscillations which occur after interruption of the primary current are assumed to have died away in a time which is small compared with the interval between break and make. In experiments on the same magneto as used previously, it was found that the ratio of the first maximum of the voltage wave to the first minimum was 1·8 with the condenser and 2·5 without it. Thus, assuming a mean frequency of 5,000 $\sim$ per second, the oscillations would have been reduced to 1 per cent. of their original value after 8×10^{-4} sec. with and 6×10^{-4} sec. without the condenser. At 1,800 revs. per min. the time between break and make is about 5×10^{-3} sec.,

so that the above assumption is justifiable. These oscillations will have an effect on the magnetic behaviour of the iron.

Since the induced current and voltage waves in the magneto, *before break*, are not sinusoidal, and since the permeability of the iron is not constant, the values of the primary and secondary inductances and effective resistances cannot be considered constant. Owing to the fact that the condenser effect is small, it is sufficiently accurate to assume that the inductances and effective resistances are constant and have the same values as those measured at the equivalent pulsatance ω, in this case about $1{,}500 \div 250 \sim$ per second, or four times the frequency of the flux.

Between break and make there is a condenser in both the primary and secondary circuits. The capacity reactance is very large compared with the resistance, and the latter can therefore be neglected. After make, there is no capacity in the primary circuit, but the effective resistance is comparable with the inductive reactance, so that it cannot be neglected.

Let L_1 L_2=effective inductance of primary and secondary windings, assumed constant;

M=mutual inductance, assumed constant, also that $L_{21}=L_{12}$;

k=coefficient of coupling;

q_1 and q_2=quantities of electricity in primary and secondary condensers;

i_1 and i_2=currents in primary and secondary circuits;

C_1 C_2=capacities in primary and secondary circuits;

R_1=effective resistance of primary winding, assumed constant;

$\frac{E_1}{E_2} \div \frac{n_1}{n_2}$=ratio of turns on primary and secondary windings.

Considering the rotational effect between break and make, after the main high frequency oscillations due to the cessation of the primary current have died away, we have for the primary circuit

$$L_1 D^2 q_1 + M D^2 q_2 + \frac{q_1}{C_1} = E_1 \sin \omega t \quad . \quad . \quad . \quad (1)$$

where $e_1 = E_1 \sin \omega t$ is the E.M.F. induced in the primary due to rotation and $D = d/dt$.

For the secondary circuit the equation is similar

$$L_2D^2q_2+MD^2q_1+\frac{q_2}{C_2}=E_2\sin\omega t. \quad . \quad . \quad . \quad (2)$$

The arrangement of the circuits is shown in Fig. 7.*

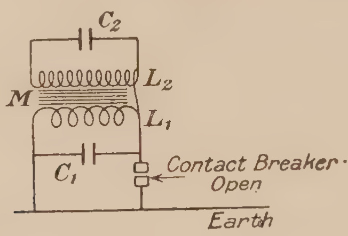


Fig. 7.

Solving these equations we obtain the primary current,

$$i_1=\frac{\frac{E_1}{\omega L_1}\left[\frac{\omega^2}{L_2C_2}+\omega^4\left(\frac{M}{L_2}\cdot\frac{n_2}{n_1}-1\right)\right]\cos\omega t}{\left(1-\frac{M^2}{L_1L_2}\right)\omega^4-\left(\frac{1}{L_1C_1}+\frac{1}{L_2C_2}\right)\omega^2+\frac{1}{L_1L_2C_1C_2}}+$$

a transient which dies away rapidly, and can therefore be neglected . . . . (3)

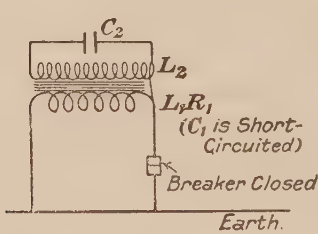


Fig. 8.

When the contacts close equation (2) remains unaltered, but (1) becomes (*see* Fig. 8)

$$L_1D^2q_1+MD^2q_2+R_1Dq_1=E_1\sin\omega t \quad . \quad . \quad . \quad (4)$$

Solving (2) and (4) we obtain

$$i_1=\text{const.}\times e^{-\frac{R_1}{L_1}t}+\frac{F\omega[G\cos\omega t+\omega H\sin\omega t]}{H^2+\omega^2G^2}+$$

* The analysis has been worked out on the assumption that the primary and secondary circuits are coupled, but not directly connected, as shown in Fig. 7. This does not introduce appreciable errors in (7).

a transient which can be neglected, since it will have died away long before break . . . . (5);
where

$$F=\frac{E_1}{L_1}\left[\frac{1}{L_2C_2}+\omega^2\left\{k\cdot\frac{n_2}{n_1}\left(\frac{L_1}{L_2}\right)^{\frac{1}{2}}-1\right\}\right];$$

$$G=\omega^2\left[\omega^2(1-k^2)-\frac{1}{L_2C_2}\right];$$

$$H=\frac{R_1}{L_1}\left(\frac{1}{L_2C_2}-\omega^2\right);$$

$$\text{Const}=\frac{-F\omega G}{G^2+\omega^2H^2};$$

since $i_1 \doteqdot 0$, when $\omega t=0$, for its value from (3) is very small.

If $C_2=0$, *i.e.*, there is no secondary condenser, and $k=0$, we obtain from (5)

$$i_1=\frac{E_1}{(R_1{}^2+\omega^2L_1{}^2)^{\frac{1}{2}}}[e^{-\frac{R_1}{L_1}t}\sin\theta+\sin(\omega t-\theta)], \quad . \quad (6)$$

where $\theta=\tan^{-1}\omega L_1/R_1$; as we should expect, since the primary has then only resistance and inductance, and the secondary is inoperative.

Thus, since the self-capacity of the secondary is small, the effect of the condenser on the primary current is obtained by subtracting (6) from (5).

The following numerical values for the constants of the primary and secondary circuits will be taken :—

$L_1=5\times10^{-3}$ henry,
$L_2=25$ henries,
$C_2=6\times10^{-10}$ farad (=self capacity of secondary+condenser)
$R_1=2$ ohms,
$\omega=1{,}500$ or $f=250$ ∼per sec.,
$n_2/n_1=65$,
$k^2=0{\cdot}9$.

Substituting these values in the expressions for F, G and H we find that the quantities

$$\omega^2\left[k\frac{n_2}{n_1}\left(\frac{L_1}{L_2}\right)-1\right];\ \omega^2(1-k^2);\ \omega^2,$$

can be neglected. Thus we can write

$$F=\frac{E_1}{L_1L_2C_2},\quad G=\frac{-\omega^2}{L_2C_2}\quad\text{and}\quad H=\frac{R_1}{L_1L_2C_2}.$$

Hence, substituting these values in (6) we obtain,

$$i_1 \doteqdot \frac{E_1}{(R_1^2+\omega^2L_1^2)^{\frac{1}{2}}}\left[e^{-\frac{R_1}{L_1}t}\sin\theta+\sin(\omega t-\theta)\right]. \quad . \quad (7)$$

Since (6) and (7) are very nearly equal, it follows that a condenser of 600 picofarads across the secondary circuit has no appreciable effect on the primary current at break. The same reasoning holds if $i_1=0$ when $\omega t=\alpha$, and break takes place when $\omega t=\alpha+\pi/2$. In either case the effect of the condenser is less than 1 per cent., assuming no dielectric loss.

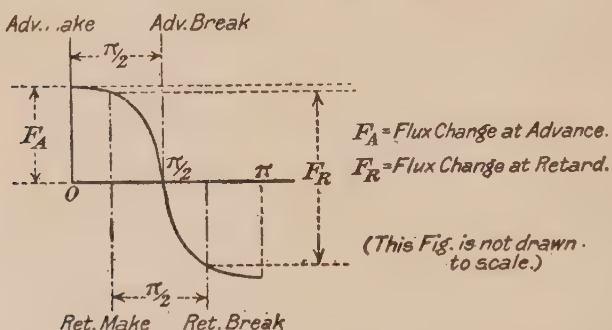


Fig. 9.—Diagram showing Flux Change in Magneto at Full Advance and at Full Retard.

The diagram is only an approximate representation of the flux changes at advance and retard. The actual slope at $\pi/2$ is greater than that shown, and when advance break occurs the flux is negative.

The relative displacements of make and break on the time axis at full advance and full retard for a B.T.-H. inductor type magneto are shown in Fig. 9.

Since the foregoing was written some experiments have been carried out with a Thomson-Bennett E.4 rotating armature magneto. In this machine the peak voltage and, therefore, the primary current, is greater at all speeds on advance than on retard. This is due partly to the design of the cam for operating the contact breaker, since it controls the relative positions of make and break, and also to the setting of the

timing lever. The capacities of the condensers necessary to reduce the peak voltage below that to cause sparking at the safety gap at 1,800 revs. per min., were 1,800 picofarads on full retard and 3,800 picofarads on full advance. Both of these values are greatly in excess of that used with the B.T.-H. machine.* The influence of the condenser on the primary current would be greater at advance than at retard, its magnitude being from 3 to 4 per cent. at 1,800 revs. per min. With the B.T.-H. inductor type magneto a condenser of 3,800 picofarads would have a greater effect, since there are two alternations of flux per revolution, whereas in the Thomson-Bennett there is only one flux alternation per revolution.

With the Thomson-Bennett machine the open circuit voltage was greater than with the B.T.-H. On full advance, where the rotational effect is greatest—assuming the same phase relations and wave form as those of Fig. 2†—the voltage due to rotation is 20 per cent. of the total peak voltage at 2,000 revs. per min., if the latter is controlled by the safety gap. A correction of this magnitude is too large. In general it is not of great importance to know the value of the primary current at speeds above 1,000 revs. per min. If, however, it is necessary to exceed this speed of rotation, some method of reducing the proportionate effect of the rotational voltage must be adopted. This can be accomplished in two ways : (*a*) using an armature with a reduced secondary in which case no condenser is required ; (*b*) cutting the safety gap out of action, using an electrostatic voltmeter reading to 15,000 volts‡ and reducing the size of the auxiliary condenser. In either of these cases the machine has to be partially dismantled, which is rather a laborious process if method (*a*) is adopted.

ABSTRACT.

A method of obtaining experimentally the current at break in a magneto is described. A condenser is connected across the secondary winding to reduce the voltage below that to cause sparking at the safety gap. The peak voltage due solely to interruption of the current at any speed is found. The interrupted direct current necessary to give the same peak voltage is also found by using a calibrating circuit.

* As indicated above the magnet of this machine was not fully magnetised. The designs of the two types of machine, however, are quite different.

† It is advisable, of course, to determine the secondary open circuit voltages at retard and advance for the particular machine under test.

‡ This is about the voltage at which a rotating armature magneto sparks across at the slip-ring when the safety gap is removed.

The magnitude of this current is equal to that broken in the magneto. The influence of the secondary condenser on the primary current at high speeds is discussed.

DISCUSSION.

Dr. Norman Campbell (communicated the following): It is quite possible to make satisfactory measurements on the primary current of a magneto by inserting in the circuit a small non-inductive resistance and taking the potential difference across it by means of a rotating "contactor." The total resistance inserted need only be 0·03 ohm; 0·01 for the leads and 0·02 for the measuring resistance. By the use of a suitable potentiometer method a change of 0·02 amp. can then be detected, which is ample for practical purposes. It was found that on increasing the total resistance inserted to 0·11 ohm, the current did not change by as much as 3 per cent.; accordingly the disturbance caused by the resistance of 0·03 ohm must be quite inappreciable.

With this arrangement it is possible to test the accuracy of Dr. McLachlan's method by examining whether the secondary peak potential is proportional to the primary current. It was found that it was very accurately proportional when the current was changed by varying the strength of the magnets. By changing that strength within wide limits it seems possible to change all currents and potentials in the circuits without changing appreciably their ratio or wave form.

On the other hand, considerable discrepancies, amounting sometimes to 10 per cent., were found when the secondary peak potential produced by breaking a primary current generated by rotation of the armature was compared with that produced by breaking the same current supplied by a battery. My experience indicates that errors of this order might occur in using Dr. McLachlan's method (allowance, of course, was made for the E.M.F. generated directly in the secondary by rotation of the armature).

For the rest, my experiments agreed in general with those of Dr. McLachlan. Since I have not kept the records I cannot discuss them in detail; but I remember being surprised how different was the open-circuit voltage wave which I found from that shown in Fig. 2.

Dr. Eccles said that Dr. McLachlan had used an indirect method and then made a series of check experiments to show that the various errors introduced are negligible. Dr. Campbell had mentioned a direct method. He thought that direct methods were preferable where possible; and suggested that the Dufour cathode ray oscillograph would have been the ideal instrument for the investigation of the magneto.

Dr. Rayner said there was one point about which he felt doubtful in the direct current calibration. The calibrating flux was not employed in the same way as the working flux, one being reversed and the other not. He thought some method might be devised to avoid this discrepancy. He was afraid the power required to work the Dufour oscillograph might be greater than was readily obtained from a magneto.

Dr. Eccles: An amplifier could be used.

Prof. Lees pointed out that the important thing in these researches was that we were obtaining knowledge about the magneto, which has hitherto been treated almost wholly on empirical lines.

The Author, in reply, stated that he had tried the method outlined by Dr. Campbell, but without the use of a rotating contactor. The results were unsatisfactory owing to the effect of transients before and after break. In order to eliminate these, it is essential that the contactor circuit should be broken before the primary. This, however, does not eliminate transients *before* break. The effect of these will be evident from the remarks in the Paper on the auxiliary coil method. Unless the interval between the breaking of the two circuits is small, errors can easily arise. Dr. Campbell's statement regarding resistance only applies at high speeds ‹ ‘, say, 700 revs. per

min. or more. At low speeds, owing to the small value of the primary inductive reactance, the current is nearly proportional to the resistance. This will be seen from Fig. 4, since the current-speed relation is linear, and the E.M.F. is proportional to the speed. Hence, increasing the resistance to 0·03 or 0·11 ohm, as Dr. Campbell did, would entail a reduction in current of 6 and 22 per cent. respectively on the initial portions of the curves of Fig. 4.

If the method of measuring the voltage drop across the resistance is one which gives the peak value, the current at break is measured only when the maximum value does not occur *before* break. If the maximum occurs before break, as it does at retard, the peak voltage across the resistance is not a measure of the current at break, although the values so obtained may be proportional to the secondary voltage of the magneto found by altering the strength of the magnet. The question arises as to whether the peak voltage—due solely to interruption, as found by Dr. Campbell's method—is proportional to the current broken when the *speed* is varied.

There are several ways in which discrepancies may arise in measuring the peak voltage by breaking a direct current in a magneto. (1) If the direction of the current is different from that during rotation, errors of the order mentioned by Dr. Campbell may occur. This can be tested by moving the armature through 180 electrical degrees and repeating the experiments. (2) The position of the armature at break must be found fairly accurately, since the polarisation of the core due to the magnet varies with the armature position. Polarisation affects the primary and secondary effective resistances and conductances, also the flux change due to a given current. These react on the peak voltage. (3) The peak voltage found by breaking a direct current (flowing in the proper direction) at advance or retard is not the same for all armature positions. This is probably due to some asymmetry. (4) The rectifying valve should pass no reverse current, and at low armature speeds the leakage should be extremely small. (5) Great care must be taken to prevent sparking at the separate contact-breaker used for calibration. If sparking, or rather arcing, occurs, there are variations in the peak voltage, and the battery and rheostat form part of the oscillatory circuit.

Unfortunately, Dr. Campbell does not state (*a*) the type of magneto, (*b*) the values of the currents, (*c*) the speeds, (*d*) the position of the timing lever, (*e*) whether the 10 per cent. error was in excess or defect. In practice one is generally concerned with currents of 2 amperes or more, and there does not seem to be any definite reason why the method given in the Paper should be subject to errors of 10 per cent. in measuring currents of this magnitude if the precautions outlined above are taken.

Dr. Eccles' suggestion to use the Dufour oscillograph for measurements on the magneto was contemplated eight months ago. Up to the present, however, it has not been possible to secure one of these instruments. Without oscillograms of the voltage wave form obtained by apparatus of this nature it is impossible to know precisely what happens in the secondary of a magneto after break. With regard to Dr. Eccles' reference to the use of an amplifier to measure the primary current, difficulties may arise due to transients which would influence the valve circuits. Owing to the fact that the magneto radiates oscillations of very short wave length (10 to 30 metres), a further source of trouble with the amplifier might arise. This, however, is a matter for experiment.

The point raised by Dr. Rayner, and mentioned in the Paper, should not have serious consequences for currents of 2 amperes or more. In view of the fact that an appreciable variation exists between the performances of magnetos of the same type, it is not necessary to measure the current more accurately than 5 per cent. This will be clear when one remembers that only about 30 per cent. of the prmary electromagnetic energy in the form $\frac{1}{2}LI^2$ is transformed into electrostatic energy in the secondary. The chief object in the determination of the primary current was to obtain an estimate of the value of $\frac{1}{2}LI^2$.

VI. *Note on a Modified Form of Wehnelt Interrupter. By*

F. H. NEWMAN, *A.R.C.Sc., B.Sc., University College, Exeter.*

RECEIVED OCT. 7, 1919.

THE Wehnelt interrupter is not used generally owing to the rapid disintegration of the platinum wire, and also because the current required for the working of the cell is large. If, however, a modified form is adopted these disadvantages disappear. A plate of aluminium, area 50 sq. cm., takes the place of the lead plate, the other electrode being platinum wire sealed in a glass tube. These electrodes are immersed in a saturated solution of ammonium phosphate made alkaline with ammonia. This interrupter works equally well with direct or alternating currents. In the former case the wire must be made the positive electrode, while with alternating currents the passage of the current is such that the wire is positive to the aluminium. The cell works with a minimum potential difference of 18 volts, but the frequency of the interruptions increases with the applied potential difference. The current produced is intermittent, and depends on the surface area of the wire, becoming greater as the surface area increases. Thus, with a wire 5 mm. long and 0·3 mm. diameter, the current with a P.D. of 30 volts is 1·0 ampere, which increases to 1·4 amperes, when the P.D. across the terminals is 100 volts. With a wire 20 mm. and 0·3 mm. diameter the current is 3 amperes for a P.D. of 35 volts. Working with a 6 in. induction coil the current in the secondary is very steady and unidirectional, and can be used for X-rays or where a steady unidirectional discharge is required for any length of time. The interrupter with 5 mm. of wire has worked continuously for 10 hours without much disintegration of the platinum wire or heating of the liquid. The ordinary Wehnelt interrupter could not be used for such a long period. It was placed directly across the mains of 100 volts alternating current, and connected in series with the primary of the induction coil. No condenser nor make-and-break is required, and the great advantage is the small current necessary for the working of the interrupter, and yet the secondary discharge is quite steady and of high tension. For the production of unidirectional secondary currents this interrupter works very well, and requires little attention.

DISCUSSION.

Capt. C. E. S. PHILLIPS said it was important that the experiments should be repeated on the scale required for ordinary X-ray work. He would be glad to afford the author any facilities he required for this purpose. Had he tried the use of potassium carbonate instead of ammonium phosphate ? In the X-ray service in the Army a reliable portable interrupter was badly required. The objection to the ordinary Wehnelt was that a voltage of 80 to 100 was required.

Mr. W. R. COOPER asked if rectification was essential to the working of the instrument. Would plates of other metals than aluminium work ?

Mr. NEWMAN said he had not tried other salts. Any metal would serve, but aluminium was best and required the least voltage.

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